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The Probability Flow in the Stock Market and Spontaneous Symmetry Breaking in Quantum Finance

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Abstract: The spontaneous symmetry breaking phenomena applied to Quantum Finance considers that the martingale state in the stock market corresponds to a ground (vacuum) state if we express the financial equations in the Hamiltonian form. The original analysis for this phenomena completely ignores the kinetic terms in the neighborhood of the minimal of the potential terms. This is correct in most of the cases. However, when we deal with the martingale condition, it comes out that the kinetic terms can also behave as potential terms and then reproduce a shift on the effective location of the vacuum (martingale). In this paper, we analyze the effective symmetry breaking patterns and the connected vacuum degeneracy for these special circumstances. Within the same scenario, we analyze the connection between the flow of information and the multiplicity of martingale states, providing in this way powerful tools for analyzing the dynamic of the stock markets.

Keywords: martingale condition; vacuum condition; spontaneous symmetry breaking; degenerate vacuum; flow of information; Hermiticity; random fluctuations; conservation of the information

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1. Introduction

When we express the financial equations of the stock market in the Hamiltonian form, the flow of information through the market can be quantified by analyzing the flow of probability. The financial Hamiltonians are in general non-Hermitian, and then they do not preserve the information. However, under some special combination of the free-parameters of the market, the financial Hamiltonian can become Hermitian, preserving the information (probability). The flow of information in the market is connected with the definition of the martingale, which is the equilibrium state where there is no flow of information. In probability theory, a martingale process is the one where no change on the condition of the system should be expected. This means that the present expectation value of a random variable is the most probable one for a future result of the same variable [1,2]. Therefore, we can imagine the martingale condition in Quantum Finance as a risk-neutral evolution of the system [3–5]. For these types of evolution, there is no chance of having arbitrage [3,6]. When there is arbitrage in finance, investors can operate simultaneously in different markets [6]. Then, for example, it might be possible for a broker to buy shares in London and then sell them in Hong Kong. The broker will then generate income from the differences in prices in both markets, over the same product. If the possibility of arbitrage exists, then this means that the market is outside equilibrium (not in the martingale state). Additionally, arbitrage is normally used by big organizations and not individuals, because only high volumes of trade can generate a considerable income after subtracting the charges due to transfers [6]. The arbitrage itself is a process which helps the financial system to distribute its information such that at the end an equilibrium condition or martingale state is obtained. When the equilibrium condition appears, then the system will follow a martingale process. The existence of a martingale condition is known as **the fundamental theorem of finance** [6,7]. In a previous paper, some of the authors formulated the spontaneous symmetry breaking in quantum finance [8]. Within this formalism, the martingale condition (state) appears as a vacuum condition which becomes degenerate under some circumstances [6,8]. In the most general sense, we have a multiplicity of vacuum states for the Black–Scholes (BS) and the Merton–Garman (MG) cases. This vacuum degeneracy is related to the symmetries under change of prices and the symmetry under changes in volatility for the MG case. Therefore, we cannot conclusively say that these symmetries are spontaneously broken [9,10] (their generators do not annihilate the vacuum state). The perfect vacuum condition is only recovered for some combination of parameters, for which the ground state (martingale state) is unique. For the regime analyzed in [8], it was possible to find a natural connection between the flow of information in the system and spontaneous symmetry breaking. Some suspects about this connection were mentioned in [11]. In [8], an extended version of the martingale state was also proposed, which includes not only prices but also the volatility as a variable. A degenerate vacuum condition again appears in these situations, with the corresponding symmetries spontaneously broken. Another interesting situation was analyzed in [8]. It corresponds to an ideal case where, for the MG and the BS case, additional non-derivative terms are included, such that the martingale condition is still satisfied. These potential terms are different to those analyzed before by some other authors [6,12]. The potential terms added in [8] are also different to the standard case studied in [13], where a double slit constraint was explored. Indeed, the potential terms analyzed in [8] correspond to collective decisions or collective behavior. Although the results obtained in [8] are correct, they were always focused on the analysis of situations, ignoring the kinetic contributions in the neighborhood of the martingale state. This corresponds to the strong-field regime to be defined in this paper. This regime, although valid, does not represent the whole scenario. For this reason, in this paper we also explore other regimes by extending the results obtained in [8]. We then explore situations where the kinetic terms behave as additional potential terms (weak-field regime and intermediate regimes). This certainly happens in reality when we consider the martingale condition in the BS equation as well as for the MG equation. Indeed, the kinetic terms cannot be ignored in general around the neighborhood of the martingale state when we consider its standard definition. Still, we can say that there are regimes where the results obtained in [8] are valid and regimes where the additional results obtained in this paper represent a more accurate picture of the reality. Finally, in this paper we fully connect the notions of spontaneous symmetry breaking and flow of information (probability) in the stock market. The paper is organized as follows: In Section 2, we explain the BS equation and we derive its Hamiltonian form. In Section 3, we explain the MG equation and again we derive its Hamiltonian. In Section 4, we illustrate the standard definition of martingale and then we justify why the martingale state (condition) can be perceived as a vacuum state in Quantum Finance. In Section 5, we evaluate the conditions under which additional terms in the potential can be included, such that the martingale condition is still preserved. In Section 6, we make some extensions of the results obtained in [8], in connection with the spontaneous symmetry breaking under changes of prices and volatility. In this paper we analyze regimes which were ignored in [8]. This means that in this paper we explore both weak and strong regimes for the quantum field, representing the series expansion of the martingale state. The martingale state then corresponds to the vacuum condition for this quantum field, denoted by ϕ_{vac}. Its explicit result depends on the regime under analysis, as well as the order of the series expansion when we express the Hamiltonian as a function of quantum fields. In Section 7, we analyze the details about the extended martingale condition, which depends not only on the prices of the options but also on the stochastic volatility. This is the section where the explicit results for the MG case are analyzed. Finally, in Section 8, we conclude.

2. The Black–Scholes Equation

The stock price $S(t)$ is normally taken as a random stochastic variable, evolving in agreement to a stochastic differential equation given by

$$\frac{dS(t)}{dt} = \phi S(t) + \sigma S R(t). \tag{1}$$

Here, ϕ is the expected return of the security, $R(t)$ is the Gaussian white noise with zero mean, and σ is the volatility [6]. Note that this simple equation contains one derivative term on the left-hand side and non-derivative terms on the right-hand side. The fundamental analysis of Black and Scholes excludes the volatility so that we can guarantee the evolution of the price of the stock with certainty [14]. In this way, by imposing $\sigma = 0$, we obtain a simple solution for the Equation (1) as

$$S(t) = e^{\phi t} S(0). \tag{2}$$

The possibility of arbitrage is excluded if we can make a perfect hedged portfolio. In this sense, any possibility of uncertainty is excluded and we can analyze the evolution of the price free of any white noise [6]. We can consider the following portfolio

$$\Pi = \psi - \frac{\partial \psi}{\partial S} S. \tag{3}$$

This is a portfolio where an investor holds the option and then short sells the amount $\frac{\partial \psi}{\partial S}$ for the security S. By using the Ito calculus (stochastic calculus) [6], it is possible to demonstrate that

$$\frac{d\Pi}{dt} = \frac{\partial \psi}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \psi}{\partial S^2}. \tag{4}$$

Deeper analysis about the Ito's formula has been done in [15]. In Equation (4), the change in the value of Π does not have any uncertainty associated with it [6]. The random term has disappeared due to the choice of portfolio. Since here we have a risk-free rate of return for this case (no arbitrage) [16,17], then the following equation is satisfied

$$\frac{d\Pi}{dt} = r\Pi. \tag{5}$$

If we use the results (3) and (4), together with the previous equation, then we get

$$\frac{\partial \psi}{\partial t} + rS\frac{\partial \psi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \psi}{\partial S^2} = r\psi. \tag{6}$$

This is the Black–Scholes equation [14,18,19], which is independent of the expectations of the investors, defined by the parameter ϕ, which appears in Equation (1). In other words, in the Black–Scholes equation, the security (derivative) price is based on a risk-free process. The basic assumptions of the Black–Scholes equation are:

(1) The spot interest rate r is constant.
(2) In order to create the hedged portfolio Π, the stock is infinitely divisible, and in addition it is possible to short sell the stock.
(3) The portfolio satisfies the no-arbitrage condition.
(4) The portfolio Π can be re-balanced continuously.
(5) There is no fee for transaction.
(6) The stock price has a continuous evolution.

Black–Scholes Hamiltonian Formulation

We will explain how the Equation (6) can be expressed as an eigenvalue problem after a change of variable. The resulting equation will be the Schrödinger equation with a non-Hermitian Hamiltonian. For Equation (6), consider the change of variable $S = e^x$, where $-\infty < x < \infty$. In this way, the BS equation becomes

$$\frac{\partial \psi}{\partial t} = \hat{H}_{BS}\psi, \tag{7}$$

where we have defined the operator

$$\hat{H}_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r. \tag{8}$$

as the BS Hamiltonian. Quantum extensions to the classical BS equation analyzed here have been proposed in [20–22]. Note that the resulting Hamiltonian in Equation (8) is non-Hermitian, since $\hat{H} \neq \hat{H}^+$ [12,23,24]. In addition, note that since the spot interest rate r is constant, then the potential term is just a constant term. This means that the vacuum condition is trivial for this case. Under the BS Hamiltonian, the evolution in time of the option is non-unitary in general (in addition, the Hamiltonian non-necessarily obeys the PT symmetry). This means that the probability is not necessarily preserved in time, although it is certainly well-defined and its total value is equal to one. In general, there are some cases in ordinary Quantum Mechanics, as well as in Quantum Field Theory, where it is interesting to explore non-Hermitian Hamiltonians (Lagrangians) [25]. Based on the previous explanations, we cannot expect the financial market to obey unitarity. The reason for this is simply because the market is not a closed system and there are many external factors influencing its behavior: for example, it is the amount of people and organizations trading at some instant of time. Then the assumption of unitarity makes no sense at all, and the Hamiltonian must be non-Hermitian. In this paper, however, when we add some potential terms to the BS and MG equations, we will impose the Hermiticity condition on them. When the symmetry is spontaneously broken and we are working at the vacuum level, all the terms in the original BS equation become irrelevant, giving then importance to (only) the potential terms. In this way, we can follow the standard formalism suggested in [25]. The same conclusions apply to any other equation only containing kinetic terms, as it is the case of the Merton–Garman (MG) equation to be analyzed shortly. Inside the MG case, the symmetry breaking process is more interesting than in the case where the same process occurs in the BS scenario. We will return back to this argument later.

3. The Merton–Garman Equation: Preliminaries and Derivation

We can consider a more general case where the security and the volatility are both stochastic. In such a case, the market is incomplete [6]. Although several stochastic processes have been considered for modeling the case with stochastic volatility [26–32], here we consider the generic case, defined by the set of equations [6]

$$\begin{aligned} \frac{dS}{dt} &= \phi S dt + S\sqrt{V}R_1 \\ \frac{dV}{dt} &= \lambda + \mu V + \zeta V^{\alpha} R_2. \end{aligned} \tag{9}$$

Here, the volatility is defined through the variable $V = \sigma^2$, and ϕ, λ, μ, and ζ are constants [16]. The Gaussian noises R_1 and R_2, corresponding to each of the variables under analysis, are correlated in the following form

$$\langle R_1(t')R_1(t)\rangle = \langle R_2(t')R_2(t)\rangle = \delta(t-t') = \frac{1}{\rho}\langle R_1(t)R_2(t')\rangle. \tag{10}$$

Here, $-1 \leq \rho \leq 1$, and the brackets $\langle AB\rangle$ correspond to the correlation between A and B. If we consider a function f, depending on the stock price, the time, as well as on the white noises, with the help of the Ito calculus, it is possible to derive the total derivative in time of this function as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \phi S \frac{\partial f}{\partial S} + (\lambda + \mu V)\frac{\partial f}{\partial V} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2} + \rho V^{1/2+\alpha}\zeta \frac{\partial^2 f}{\partial S \partial V} + \frac{\zeta^2 V^{2\alpha}}{2}\frac{\partial^2 f}{\partial V^2} + \sigma S \frac{\partial f}{\partial S} R_1 + \zeta V^{\alpha} \frac{\partial f}{\partial V} R_2. \tag{11}$$

This equation can be expressed in a more compact form, which separates the stochastic terms from the non-stochastic ones as follows

$$\frac{df}{dt} = \Theta + \Xi R_1 + \psi R_2. \tag{12}$$

Here we have defined

$$\Xi = \sigma S \frac{\partial f}{\partial S}, \qquad \psi = \zeta V^{\alpha} \frac{\partial f}{\partial V},$$
$$\Theta = \frac{\partial f}{\partial t} + \phi S \frac{\partial f}{\partial S} + (\lambda + \mu V)\frac{\partial f}{\partial V} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2} + \rho V^{1/2+\alpha}\zeta \frac{\partial^2 f}{\partial S \partial V} + \frac{\zeta^2 V^{2\alpha}}{2}\frac{\partial^2 f}{\partial V^2}, \tag{13}$$

keeping in this way the notation used in [6] for convenience.

3.1. Derivation of the Merton–Garman Equation

If we consider two different options defined as C_1 and C_2 on the same underlying security with strike prices and maturities given by K_1, K_2, T_1, and T_2, respectively. It is possible to create a portfolio

$$\Pi = C_1 + \Gamma_1 C_2 + \Gamma_2 S. \tag{14}$$

If we consider the result (12), then we can define the total derivative with respect to time as for the folio as

$$\frac{d\Pi}{dt} = \Theta_1 + \Gamma_1\Theta_2 + \Gamma_2\phi S + (\Xi_1 + \Gamma_1\Xi_2 + \Gamma_2\sigma S)R_1 + (\psi_1 + \Gamma_1\psi_2)R_2. \tag{15}$$

Note that this result is obtained after recognizing $f(t) = C_1$ or $f(t) = C_2$ in Equation (12) when it corresponds. It has been demonstrated that even in this case of stochastic volatility, it is still possible to create a hedged folio and then at the end we arrive again to the condition (5), after finding special constraints for Γ_1 and Γ_2 such that the white noises are removed. The solution for Π is a non-trivial one for this case, and then it requires the definition of the parameter

$$\beta(S,V,t,r) = \frac{1}{\partial C_1/\partial V}\left(\frac{\partial C_1}{\partial t} + (\lambda + \mu V)\frac{\partial C_1}{\partial S} + \frac{VS^2}{2}\frac{\partial^2 C_1}{\partial S^2} + \rho V^{1/2+\alpha}\zeta\frac{\partial^2 C_1}{\partial S \partial V}\right) + \frac{1}{\partial C_1/\partial V}\left(\frac{\zeta^2 V^{2\alpha}}{2}\frac{\partial^2 C_1}{\partial V^2} - rC_1\right)$$
$$= \frac{1}{\partial C_2/\partial V}\left(\frac{\partial C_2}{\partial t} + (\lambda + \mu V)\frac{\partial C_2}{\partial S} + \frac{VS^2}{2}\frac{\partial^2 C_2}{\partial S^2} + \rho V^{1/2+\alpha}\zeta\frac{\partial^2 C_2}{\partial S \partial V} + \frac{\zeta^2 V^{2\alpha}}{2}\frac{\partial^2 C_2}{\partial V^2} - rC_2\right) \tag{16}$$

This parameter does not appear for the case of the BS equation. Indeed, β in the MG equation is defined as the market price volatility risk because the higher its value is, the lower the intention is of the investors to risk. Take into account that in the MG equation the volatility is a stochastic variable. Since the volatility is not traded in the market, then it is not possible to make a direct hedging process over this quantity [6]. In this way, when we have stochastic volatility, it is necessary to consider the expectations of the investors. This effect appears through the parameter β. It has been demonstrated in [33] that the value of β, in agreement with Equation (16), is a non-vanishing result. In general, it is always assumed that the risk of the market (in price) has been included inside the MG equation. The MG equation is then obtained by rewriting the Equation (16) in the form

$$\frac{\partial C}{\partial t}+rS\frac{\partial C}{\partial S}+(\lambda+\mu V)\frac{\partial C}{\partial V}+\frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2}+\rho\zeta V^{1/2+\alpha}S\frac{\partial^2 C}{\partial S\partial V}+\zeta^2V^{2\alpha}\frac{\partial^2 C}{\partial V^2}=rC, \quad (17)$$

where the effects of β now appear contained inside the modified parameter λ in this equation. In other words, we have shifted the parameter $\lambda\to\lambda-\beta$ in Equation (17). Later in this paper, we will express this equation in the Hamiltonian form, which is the ideal one for understanding the concept of spontaneous symmetry breaking in Quantum Finance.

3.2. Hamiltonian form of the Merton–Garman Equation

The previously analyzed MG equation can be formulated as a Hamiltonian (eigenvalue) equation. We can define a change of variable defined as

$$\begin{aligned} S=e^x, &\quad -\infty<x<\infty,\\ \sigma^2=V=e^y, &\quad -\infty<y<\infty, \end{aligned} \quad (18)$$

and then the MG Equation (17) becomes [6,34,35]

$$\frac{\partial C}{\partial t}+\left(r-\frac{e^y}{2}\right)\frac{\partial C}{\partial x}+\left(\lambda e^{-y}+\mu-\frac{\zeta^2}{2}e^{2y(\alpha-1)}\right)\frac{\partial C}{\partial y}+\frac{e^y}{2}\frac{\partial^2 C}{\partial x^2}+\rho\zeta e^{y(\alpha-1/2)}\frac{\partial^2 C}{\partial x\partial y}+ \zeta^2e^{2y(\alpha-1)}\frac{\partial^2 C}{\partial y^2}=rC. \quad (19)$$

If we express this equation as an eigenvalue problem in the same form as in Equation (7) for the BS case, by following the same arguments illustrated in [6,36], then we have the result

$$\frac{\partial C}{\partial t}=\hat{H}_{MG}C, \quad (20)$$

with the MG Hamiltonian defined as

$$\hat{H}_{MG}=-\frac{e^y}{2}\frac{\partial^2}{\partial x^2}-\left(r-\frac{e^y}{2}\right)\frac{\partial}{\partial x}-\left(\lambda e^{-y}+\mu-\frac{\zeta^2}{2}e^{2y(\alpha-1)}\right)\frac{\partial}{\partial y}-\rho\zeta e^{y(\alpha-1/2)}\frac{\partial^2}{\partial x\partial y}- \zeta^2e^{2y(\alpha-1)}\frac{\partial^2}{\partial y^2}+r. \quad (21)$$

Exact solutions for the MG equation have been found for the case $\alpha=1$ in [34] by using path-integral techniques. The same equation has been solved in [26–32] for the case $\alpha=1/2$ by using standard techniques of differential equations. Note that the equation has two degrees of freedom. Later, we will see that when we have spontaneous symmetry breaking, it becomes irrelevant to know the exact solution of this equation.

4. The Martingale Condition in Finance

The martingale condition is required for having a risk-neutral evolution for the price of an option. This means that the price of a financial instrument is free of any possibility of arbitrage. In probability theory, the risk-free evolution is modeled inside a stochastic process. Assume, for example, $N+1$ random variables X_i, with a joint probability distribution defined as $p(x_1,x_2,...,x_{N+1})$. Then the martingale process is simply defined as the condition under which

$$E[X_{n+1}|x_1,x_2,...,x_n]=x_n, \quad (22)$$

is satisfied [6]. Note that $E[X_i]$ is the expectation value of the random variable. Equation (22) suggests that the expected value of a subsequent observation of a random variable is simply the present value. For the purpose of this paper, the random variables correspond to the future prices of the stock given by S_1, S_2, ..., S_{N+1}, which are defined at different times

$t_1, t_2, ..., t_{N+1}$. We can then apply the same martingale condition to the stocks if we make the corresponding discounts in order to compare prices defined at different moments [3,6]. We can assume that the future value of an equity is defined as $S(t)$. If there is a free-risk evolution of the discounted price defined as

$$e^{-\int_0^t r(t')dt'} S(t). \tag{23}$$

Then the value follows the martingale process [7]. In this way, the conditional probability for the present price is the actual value given by $S(0)$. The martingale condition can then be expressed as [6]

$$S(0) = E\left[e^{-\int_0^t r(t')dt'} S(t)|S(0)\right], \tag{24}$$

and this result is general. Equivalent expressions have been used for the analysis of the evolution of forward rates [6]. The importance of martingales is analyzed in [37]. The interpretation of Equation (24) is clear. The left-hand side is just the present price of the security. The right-hand side is the expected value of the discounted price of the security at the time t. Discounted means that the quantity evaluated at the time t has to be extrapolated to the present value. Both quantities must be equivalent under the martingale condition.

The Martingale Condition as a Vacuum Condition for a Hamiltonian

Here we justify why the martingale condition can be perceived as a vacuum state from the perspective of the Hamiltonian formulation. Consider as before an option on a security $S = e^x$ that matures at time T with the corresponding pay-off function $g(x)$. In this way we can describe the risk-free evolution of the option as

$$C(t, x) = \int_{-\infty}^{\infty} dx' \langle x|e^{-(T-t)\hat{H}}|x'\rangle g(x'). \tag{25}$$

By using the previous definition of martingales, for this case we have

$$S(t) = E\left[e^{-(t_*-t)r} S(t_*)|S(t)\right]. \tag{26}$$

If we introduce $S(x)$ (the price of the security) in Equation (25), then under the martingale condition, we have

$$S(t, x) = \int_{-\infty}^{\infty} dx' \langle x|e^{-(t_*-t)\hat{H}}|x'\rangle S(x'). \tag{27}$$

This equation can be re-expressed in Dirac notation as

$$\langle x'|S\rangle = \int_{-\infty}^{\infty} dx' \langle x|e^{-(t_*-t)\hat{H}}|x'\rangle \langle x'|S\rangle. \tag{28}$$

If we take the base $|x'\rangle$ as a complete set of states, then the condition $\hat{I} = \int dx' |x'\rangle\langle x'|$ ($\hat{I}$ is the identity matrix) is satisfied and then the previous expression is simplified as

$$|S\rangle = e^{-(t_*-t)\hat{H}}|S\rangle. \tag{29}$$

Then there is no Hamiltonian (time) evolution for the state $|S\rangle$ under the previous conditions. It also comes out that the Hamiltonian annihilates the same state as follows

$$\hat{H}|S\rangle = 0. \tag{30}$$

Since the Hamiltonian annihilates the martingale state, then we can interpret it as a vacuum state. In standard Quantum Mechanics, the Hamiltonian annihilates the vacuum (ground) state, which is additionally a state representing an equilibrium condition. This is precisely what the martingale state represents in Quantum Finance, in agreement with the result (30) [38–40]. Interestingly, the BS Hamiltonian given in Equation (8), as well as the MG Hamiltonian defined in Equation (21), satisfy the martingale condition in the form defined in Equation (30).

5. Non-Derivative Terms Introduced in the Financial Hamiltonians

It is possible to introduce potential terms to the BS equation as well as to the MG one, as it is explained in [6]. It has been demonstrated that the martingale condition can still be maintained if the potential satisfies some special conditions. In general, a potential term will appear as

$$\hat{H}^{eff}_{BS,MG} = \hat{H}_{BS,MG} + \hat{V}(x), \tag{31}$$

with the potential term $\hat{V}$ containing non-derivative terms depending on the security S. Since we usually have a change of variables in the Hamiltonian formulation, this functional dependence is indirect. In the previous equation, $\hat{H}^{eff}_{BH,MG}$ is the effective Hamiltonian, including the potential contribution. Some barrier options as well as some path-dependent options admit the inclusion of potential terms for their deep understanding [12,23,24]. On the other hand, for the case of the Black–Scholes Hamiltonian, the martingale condition is maintained if the potential appears in the Hamiltonian in the following form [6]

$$\hat{H}^{eff}_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - V(x)\right)\frac{\partial}{\partial x} + V(x). \tag{32}$$

Then an effective Hamiltonian expressed in this way can be used for pricing the option. The discount in these general cases depends on the price of the option itself. Then the security discount defined in Equation (23) is modified as [6]

$$e^{-\int_0^t r(t')dt'}S(t) \to e^{-\int_0^t V(x(t'))dt'}S(t). \tag{33}$$

In [6] it is argued that the usual discounting of a security using the spot interest rate r is determined by the argument of no arbitrage involving fixed deposits in the money market account. Studies about viable potentials matching with the reality of the market are under analysis. It is important to notice that the Hamiltonian (32) can be converted to a Hermitian operator by using a similarity transformation, as has been reported in [6]. Then we can define

$$\hat{H}^{eff}_{BS} = e^s \hat{H}_{Herm} e^{-s}. \tag{34}$$

Here the Hermitian Hamiltonian is defined as

$$\hat{H}_{Herm} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}V'(x) + \frac{1}{2\sigma^2}\left(V + \frac{1}{2}\sigma^2\right)^2, \tag{35}$$

and $s = x/2 - (1/\sigma^2)\int_0^x dyV(y)$. This result can be obtained by replacing (32) and (35) in Equation (34). From the Hermitian Hamiltonians, it is possible to construct a complete basis and then we can find real eigenvalues associated to this Hamiltonian. Note in particular that in the Black–Scholes case, $V(x) = r$ is constant. It is a simple task to demonstrate that the Hermitian Hamiltonian obtained by similarity transformation can be also expressed as

$$\hat{H}_{Herm} = e^{\alpha x}\left(-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \gamma\right)e^{-\alpha x}, \tag{36}$$

with

$$\gamma = \frac{1}{2\sigma^2}\left(r + \frac{1}{2}\sigma^2\right)^2 \qquad \alpha = \frac{1}{\sigma^2}\left(\frac{1}{2}\sigma^2 - r\right). \tag{37}$$

Among the trivial examples of potentials already analyzed in the literature, we find the "Down-and-Out" barrier option, where the stock price has to be over some minimal value, below which it becomes worthless. This behavior can be guaranteed with an infinite potential barrier boundary condition imposed for the value of the corresponding price. This case can be worked out directly from the non-Hermitian Hamiltonian defined in Equation (31). Another example of potential corresponds to the "Double-Knock-Out" barrier option, where it is easier to work with the Hermitian Hamiltonian part as it is defined in Equation (34), but including the potential part as follows

$$\hat{H}_{DB} = \hat{H}_{BS} + \hat{V}(x) = e^{s}\left(\hat{H}_{Herm} + \hat{V}(x)\right)e^{-s}. \tag{38}$$

This definition is used for analyzing cases where the stock has to be maintained between a maximal and a minimal value. Note that the definition (38) can be also used for the analysis of the "Down-and-Out" barrier if we focus on the non-Hermitian part (the term in the middle of the equation) with the corresponding potential. Note that these examples of potentials representing real situations are trivial cases. More details about these examples can be found in [6].

6. Deeper Analysis for the Black–Scholes and the Merton–Garman Equation

If we analyze the martingale condition, we can notice that its interpretation as a vacuum condition is not perfect. The reason is that although the Hamiltonian annihilates the martingale state $|S\rangle$, as can be seen from Equation (30), the momentum operator corresponding to the prices of the options ($\hat{p}_x$) does not annihilate the vacuum perfectly. This can be seen from their definitions as follows

$$\hat{p}_x|S\rangle = e^{x}|S\rangle. \tag{39}$$

This means that the symmetry under translations of the prices, carried out from the security S, is spontaneously broken. An exception is the case where $S \to 0$ and then $x \to -\infty$, as can be seen from Equation (18). This, however, would give us a trivial value for the security as $S = 0$. For general values, the symmetry under translations of prices is spontaneously broken. This means that the different values of S represent different possible vacuums or ground states. Different vacuums, however, would have different amounts of information because under the ideal conditions, all the information of the market is stored inside the prices. The action of $\hat{p}_x$ over $|S\rangle$ then maps one vacuum toward another one defined through the selected value x. Such operation could be seen as a rotation in a complex plane if we make the transformation of variables $x \to in\theta$. In such a case, we have

$$\hat{p}_x|S\rangle = e^{in\theta}|S\rangle = |S'\rangle \neq |S\rangle. \tag{40}$$

Here n is just a number and θ is a dimensionless phase. After this change of variable, the action of $\hat{p}_x$ is to map one vacuum into another one through a rotation defined by the phase θ. This condition shows the vacuum degeneracy. The same condition has different meanings depending on whether we consider the BS or the MG equation. This important detail about the Martingale condition deserves more attention.

6.1. Standard Definition of Spontaneous Symmetry Breaking

In standard situations, the ground state of a system shares the same symmetries of its Hamiltonian (Lagrangian). However, under special circumstances, for some specific combinations of the free-parameters, the equilibrium configuration of the system might change, up to the point of developing a new vacuum state configuration. When spontaneous symmetry breaking occurs, the free-parameters of a system take some values for which the single vacuum state, being fully symmetric, becomes unstable [9,38–40]. In such a case, then any small fluctuation on the system forces it to select a more stable vacuum which does not respect all the symmetries of the system. The symmetries of the system which the vacuum does not satisfy are called broken symmetries [9,38–40]. In these situations we claim that some of the symmetries of the system are spontaneously broken. The most famous example of spontaneous symmetry breaking is the case of the "Mexican hat" [9,38–40]. We can take, for example, the following Lagrangian corresponding to a complex scalar field ϕ [9,38–40]

$$\pounds = \partial^\mu \phi^* \partial_\mu \phi - V(\phi^* \phi), \tag{41}$$

where the potential term is defined as

$$V(\phi^* \phi) = -\mu^2 \phi \phi^* + \lambda (\phi \phi^*)^2. \tag{42}$$

Here, $\lambda > 0$ is a free-parameter of the system as well as μ^2. The Lagrangian (41) is symmetric under the $U(1)$ symmetry. If we decompose the field ϕ as

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2). \tag{43}$$

Here we can select arbitrarily which component corresponds to the order parameter and which one corresponds to the Nambu–Goldstone boson field [9,38–40]. Here by convenience we select in the standard notation the order parameter to be

$$< 0|\phi_1|0 >= \pm \frac{\mu}{\sqrt{\lambda}}. \tag{44}$$

The corresponding Nambu–Goldstone field is just $< 0|\phi_2|0 >= 0$. No matter what the notation is, the vacuum expectation value of the field in Equation (43) is simply $< 0|\phi|0 >= \pm\mu/\sqrt{\lambda}$. Note that in this example, we focus on the Lagrangian of some specific system. However, the Hamiltonian formulation contains exactly the same information of the Lagrangian formulation. Both of them are connected through a Legendre transformation [41]. In this paper we do not analyze the role of the Nambu–Goldstone field. All we care about in this paper is the conditions under which there is spontaneous symmetry breaking in Quantum Finance and the relations between variables emerging from such conditions. In general, we can define the ground state (vacuum state) of a system as the state which is annihilated by its Hamiltonian $\hat{H}|0 >= 0$. In standard situations, all the symmetry generators of the system also annihilate the same ground state. However, when spontaneous symmetry breaking occurs, then some symmetry generators cannot annihilate the vacuum state of the system. Such generators are called broken generators and they are related to the broken symmetries of the system. Then, for example, if a system breaks its symmetry with respect to spatial translations spontaneously, this means that the momentum $\hat{p}$ (generator of spatial translations) cannot annihilate the ground state of the system, namely, $\hat{p}|0 >\neq 0$. This important statement is what we use in this paper for developing our results around the BS and the MG equations.

6.2. Reinterpretation of the Martingale Condition

Now that we understand in deep detail some aspects of the Martingale condition and the exact definition of spontaneous symmetry breaking, we can re-define the martingale state $S(x,t)$ as a quantum field by doing the following change of variable $S(x,t) = e^x = \sum_{n=0}^{\infty} x^n/n! = \sum_{n=0}^{\infty} \phi^n(x,t)$, as it was proposed in [8]. The purpose of this redefinition is to transform the BS and MG equations from differential equations to algebraic ones. This is not different in essence to what is done in Quantum Field Theory or in standard Quantum Mechanics, when we use the Fourier transformation with the same purpose. Of course, we could make further expansions in terms of annihilation and creation operators, but for the purposes of this paper, it is not necessary to make further definitions for the financial fields. Having re-defined the martingale state as a field, then we can express the BS equation as a function of $\phi(x,t)$ by replacing this definition inside Equation (8), obtaining in this way

$$\hat{H}_{BS} S(x,t) = -\frac{\sigma^2}{2} n(n-1)\phi^{n-2} + \left(\frac{1}{2}\sigma^2 - r\right) n\phi^{n-1} + r\phi^n(x,t) = V(\phi) = 0, \tag{45}$$

which is valid in the neighborhood of the martingale state (vacuum). Here we omit the sum symbol because the comparison has to be done term by term in the series. This means that we have one equation for each term in the series expansion for ϕ. In Equation (45), we can see that all the terms in the Hamiltonian can now be considered to be potential terms. Note that the result (45) is valid if $d\phi/dx = \sum_n \phi/n$. In some regimes (especially inside the weak-field approximation), the kinetic terms in Equation (45) play a fundamental role in the identification of the vacuum state. The vanishing condition in Equation (45) is a natural consequence of the martingale state definition, which, being a vacuum condition, is annihilated by the Hamiltonian. This is the case because the symmetries under time-translation are not spontaneously broken. We can solve the martingale condition in Equation (45), obtaining then

$$\phi_{vac} = \frac{n}{2}\left(1-\frac{\sigma^2}{2r}\right)\left(1\pm\sqrt{1+\frac{8r\sigma^2}{(\sigma^2-2r)^2}\frac{(n-1)}{n}}\right), \quad n\neq 0. \tag{46}$$

This is the definition of the martingale state in the sense of the quantum field ϕ, as a function of the free-parameters of the theory. Note that for $n=0$, the result (46) is not well-defined, and then we impose the condition $n\neq 0$. Another special case is $n=1$, for which Equation (46) provides the following solutions

$$\phi_{vac} = \frac{1}{2}\left(1-\frac{\sigma^2}{2r}\right)(1\pm 1). \tag{47}$$

The result for this previous equation gives a trivial non-degenerate vacuum $\phi_{vac}^{n=1}=0$, but it can also give the following non-trivial result

$$\phi_{vac}^{n=1} = 1-\frac{\sigma^2}{2r}. \tag{48}$$

Note that this result is the same obtained in [8] when the BS eq. was analyzed. This is the case because in [8], the kinetic terms were ignored (Strong-field regime), which is precisely what happens in Equation (45) when $n=1$. However, note that the result (48) appeared for $n=2$ instead of $n=1$ in [8]. The reason for this is that in [8], the derivative of the potential was considered instead of the potential itself, as it is the case here. In any case, this only demonstrates that the regimes analyzed in this paper are more general than the simple case studied in [8]. All the other results, with $n\neq 0,1$ are non-trivial. Equation (45) is by itself a potential function $V(\phi)$ and we can demonstrate that any derivative of this potential will not give any information different to the one obtained from Equation (45). Then, for example, if we find find the extreme condition for Equation (45) as $\partial V(\phi)/\partial\phi = 0$, we get

$$\phi^2+\left(\frac{1}{2}\frac{\sigma^2}{r}-1\right)(n-1)\phi-\frac{\sigma^2}{2r}(n-1)(n-2)=0, \tag{49}$$

which is just essentially the same result in Equation (45), but with the coefficient terms shifted. Then, for this case, when $n=1$, the previous equation gives a trivial result $\phi=0$. In this way, since any higher-order derivative provides the same information as the one obtained from Equation (45), for the purpose of analysis, we will just focus on Equation (45) for understanding the different regimes. Finally, note that although $\hat{H}_{BS}\phi^n = 0$, still $\hat{p}\phi^n = n\phi^{n-1}\neq 0$, considering $n\neq 0$. Then, the vacuum is only annihilated by the momentum if the field ϕ is itself trivial—something which happens for some combination of parameters. Then, for example, Equation (48) gives a trivial result when $\sigma^2=2r$, which is the same condition for the Hamiltonian to be Hermitian. Then, at least for this particular case, there is a direct connection between flow of information and spontaneous symmetry breaking. When the martingale state is unique, no random fluctuations can change the prices in the market. However, when the martingale (vacuum) state is degenerate, then a multiplicity of possible ground states emerge and then any fluctuation generates changes in the information of the system, and as a consequence, changes on the prices of the Option under analysis.

Vacuum Conditions for $n \neq 0, 1$: Weak and Strong Field Regimes

(1) **Weak-field regime:** $\phi^n << \phi$
For this regime, from Equation (45), we obtain

$$-\frac{\sigma^2}{2}(n-1) + \left(\frac{1}{2}\sigma^2 - r\right)\phi \approx 0, \tag{50}$$

which can be easily solved to obtain

$$\phi_{vac} \approx \frac{\sigma^2}{\sigma^2 - 2r}(n-1). \tag{51}$$

Note that this result is trivial for $n = 1$ for any combination of parameters. This means that in these situations, again random fluctuations cannot change the prices in the market. The case (51) is also trivial when $\sigma^2 = 0$ (zero volatility). Zero volatility means zero fluctuations and then zero possibility of changing the prices in the market.

(2) **Strong Field regime:** $\phi^n >> \phi$
In this regime, Equation (45) gives the following condition

$$r\phi^2 + \left(\frac{1}{2}\sigma^2 - r\right)n\phi \approx 0. \tag{52}$$

We have two solutions; one of them is the trivial $\phi_{vac} \approx 0$ and the second solution is

$$\phi_{vac} = \left(1 - \frac{1}{2}\frac{\sigma^2}{r}\right)n. \tag{53}$$

Here again, this result is trivial if $\sigma^2 = 2r$, which is again the condition for the Hamiltonian to be Hermitian and then preserve information. Therefore, it seems that this is the combination of parameters which avoids changes of information due to random fluctuations. Before going to the next section, we have to remark that the results obtained for the BS case will be valid for the MG case because the standard definition of martingale is independent of the stochastic volatility. The only change to be done for the solutions inside the MG equation, when we consider ϕ_{vac}, is in the definition of the volatility $\sigma^2 = e^y$, with y representing the variable connected to the stochastic volatility.

7. A More General Condition for the Symmetry Breaking in the Merton Garman Equation

When we analyze the MG equation, the martingale condition is normally taken such that it is independent on the stochastic volatility. The volatility is a function of the variable y, as it is defined in Equation (18). If the martingale condition is taken as independent of y, then any term with derivative with respect to this variable will annihilate the state $S(x,t)$ defined previously. From this perspective, y can be taken as fixed when we are determining the vacuum conditions. In this portion of the paper, we would like to define a more general martingale condition, such that the possible changes in y can be considered. We will take the martingale state as

$$\hat{H}_{MG}e^{x+y} = \hat{H}_{MG}S(x,y,t) = 0. \tag{54}$$

Here, we extend the arguments showed in Equation (18), considering the extensions of the original martingale state $S(x,t) = e^x$. The condition (54) will be considered here as the extended martingale condition with $S(x,y,t) = e^{x+y}$. By using the Hamiltonian (21) and the result (54), we can observe that the condition for the Hamiltonian to annihilate the martingale state is

$$\lambda + e^y\left(\mu + \frac{\zeta^2}{2}e^{2y(\alpha-1)} + \rho\zeta e^{y(\alpha-1/2)}\right) = 0, \tag{55}$$

as far as $e^{x+y} \neq 0$, avoiding then the trivial solution. This previous condition is necessary for the state e^{x+y} to be considered as the martingale state and it represents a constraint among the free-parameters involved in the MG equation.

7.1. The Extended Martingale Condition and the Flow of Information

Previously, when we studied the ordinary martingale condition, we could demonstrate that it can be also considered as a vacuum state. The vacuum in the BS case came out to be single if there is no flow of information through the boundaries of the system and it was degenerate if the information flows through the boundaries of the system. When the vacuum is single, the momentum, defined as the generator of the changes in prices, is a perfect symmetry. On the other hand, when the vacuum is degenerate, then the same symmetry is spontaneously broken because although the Hamiltonian annihilates the ground state (martingale condition), the momentum does not do it. The interesting point here is the connection between spontaneous symmetry breaking and flow of information, which is connected with the changes on the prices in the market [6]—something which has been suggested before in [11] in a different context. By using the previously proposed extended martingale state, we can express it as a function of quantum fields in the form $S(x,y,t) = e^{x+y} = (\sum_{n=0}^{\infty}\phi_x^n)\left(\sum_{m=0}^{\infty}\phi_y^m\right)$. By replacing this expression in Equation (21), we get

$$-\frac{e^y}{2}n(n-1)\phi_x^{n-2}\phi_y^m - \left(r - \frac{e^y}{2}\right)n\phi_x^{n-1}\phi_y^m - \left(\lambda e^{-y} + \mu - \frac{\zeta^2}{2}e^{2y(\alpha-1)}\right)m\phi_x^n\phi_y^{m-1}$$
$$-\rho\zeta e^{y(\alpha-1/2)}nm\phi_x^{n-1}\phi_y^{m-1} - \zeta^2 e^{2y(\alpha-1)}m(m-1)\phi_x^n\phi_y^{m-2} + r\phi_x^n\phi_y^m = 0. \tag{56}$$

The vanishing condition comes out from the definition of martingale. The previous result is valid if $d\phi_x/dx = \sum_n \phi_x/n$ and $d\phi_y/dy = \sum_n \phi_y/n$. The Equation (56) is a quadratic equation for both fields, namely, ϕ_x and ϕ_y. Therefore, we can decide to solve it with respect to either field if convenient. In this paper we will not do it; we will rather analyze a few interesting cases and regimes in order to visualize the structure of the extended martingale condition for the MG equation. With the new definition of martingale, it is known that the symmetries under changes of price as well as the symmetries under changes of volatility are both spontaneously broken. This means that the information of the system changes due to fluctuations in the prices as well as fluctuations on the volatility. These situations are represented Mathematically as [8]

$$\hat{p}_x|S\rangle \neq 0, \qquad \hat{p}_y|S\rangle \neq 0. \tag{57}$$

with $|S\rangle$ representing the ket related to the martingale state $S(x,y,t)$. Additionally, $\langle x,y|\hat{p}_x|S\rangle = \partial_x S(x,y,t)$ and $\langle x,y|\hat{p}_y|S\rangle = \partial_y S(x,y,t)$; representing in this way the action of the operators $\hat{p}_{x,y}$ as the generators of translations in the prices and volatility related to the options. We can now analyze some important cases related to the condition (56). Note, for example, that for $n = m = 0$, we do not get anything interesting from this equation. Indeed, for this case, $r = 0$ (zero interest rate). On the other hand, we can analyze the following interesting cases

7.1.1. Extended Martingale with $n = 0$ and $m = 1$

For this case, Equation (56) gives us the following result

$$\phi_{y\ vac} = \frac{1}{r}\left(\lambda e^{-y} + \mu - \frac{\zeta^2}{2}e^{2y(\alpha-1)}\right), \tag{58}$$

with an arbitrary value for $\phi_{x\,vac}$. This means that for this particular case, the symmetry under changes in the volatility is spontaneously broken, except when the condition

$$\lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)} = 0, \tag{59}$$

is satisfied. This condition, complemented with Equation (55), gives the result

$$e^{y(\alpha-3/2)} = -\frac{\rho}{\zeta}, \tag{60}$$

which is a constraint related to important parameters connected to the stochastic volatility [6]. Note that in this case, the field ϕ_x is arbitrary. This means that the symmetry under changes of the prices is also spontaneously broken. Note, in addition, that the condition (59) corresponds to the condition for recovering the Hermiticity with respect to the variable y. As a consequence of this, the unique vacuum condition with respect to y, here represented by $\phi_{y\,vac} = 0$, is connected to the no-flow of information with respect to volatility variable y. This means that when (59) is valid, then the random fluctuations coming from the stochastic volatility do not affect do not produce changes on the information of the system. Still for this case however, the information is not completely preserved due to the random fluctuations on the prices, which disappear when $r = e^y/2$, eliminating in this way all the non-Hermiticities in the MG Hamiltonian in Equation (21).

7.1.2. Extended Martingale with $n = 1$ and $m = 0$

For this case, Equation (56) gives us the solution

$$\phi_{x\,vac} = 1 - \frac{e^y}{2r}, \tag{61}$$

with ϕ_y arbitrary. Compare Equation (61) with the case $n = 1$ of the BS result in Equation (48). The result is basically the same, but with the re-defined volatility as a stochastic variable $\sigma^2 = e^y$. Here again, the single vacuum condition $\phi_{x\,vac} = 0$ is related to the no-flow of information, but this time with respect to the variable x which is a function of the price of the stock. For this case, $\phi_{y\,vac}$ is arbitrary and then still the symmetries with respect to the changes in volatility are spontaneously broken. In this way, only the random fluctuations of the volatility affect the information content of the market.

7.1.3. Extended martingale with $n = 1$ and $m = 1$

For this case, Equation (56) gives us the solution

$$-\left(r - \frac{e^y}{2}\right)\phi_y - \left(\lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)}\right)\phi_x - \rho\zeta e^{y(\alpha-1/2)} + r\phi_x\phi_y = 0. \tag{62}$$

We can solve this expression for either quantum field, ϕ_y or ϕ_x. One interesting situation emerges when the Hermiticity condition for the Hamiltonian (21) is satisfied. This occurs when Equation (59) is satisfied and when $e^y = 2r$. In such a case, we have from Equation (62)

$$\phi_{x\,vac}\phi_{y\,vac} = \frac{\rho\zeta e^{y(\alpha-1/2)}}{r}. \tag{63}$$

If, in addition $\rho = 0$ or $\zeta = 0$, then we would have $\phi_x\phi_y = 0$, and then either the symmetry is spontaneously broken with respect to translation in prices but not with respect the variations in volatility, or on the contrary, the symmetry is spontaneously broken with respect to changes in the volatility but we still have a perfect symmetry with respect to changes in the prices of the options. The same result also suggests that the simultaneous changes in the price with a subsequent changes in volatility (and viceversa) represent a perfect symmetry for the system as far as $\rho = 0$ or $\zeta = 0$. This means that for this combined symmetry, simultaneous random fluctuations of the volatility and prices keep the information of the system unchanged. Note that ρ is a variable representing the correlation between the Gaussian noises R_1 and R_2, as it was defined in Equation (10). Therefore, $\rho = 0$ means zero correlation between the Gaussian noises. On the other hand, if $\zeta = 0$, then the Gaussian noise R_2 decouples and it comes out that it does not affect the changes in the stochastic volatility. This last issue can be observed from Equation (9).

7.1.4. Strong Field Condition $\phi_x^n >> \phi_x$ and $\phi_y^m >> \phi_y$; for Any Value of n, m

For this case, from Equation (56), we obtain

$$\phi_{x\ vac}\phi_{y\ vac} \approx \left(1 - \frac{e^y}{2r}\right) n\phi_{y\ vac} + \left(\frac{\lambda}{r}e^{-y} + \frac{\mu}{r} - \frac{\zeta^2}{2r}e^{2y(\alpha-1)}\right) m\phi_{x\ vac}, \tag{64}$$

which can be approximated to zero under the weak-field approximation if we ignore cubic order terms in Equation (56) in comparison to the fourth-order terms. Even if we do not ignore such terms, Equation (64) is exactly zero under the Hermiticity conditions (no-flow of information) defined by Equation (59) and the constraint $e^y = 2r$. In any case, this result shows that for strong field approximation for both fields, connected to the symmetries related to changes in prices and volatility, it is not possible to have a single vacuum condition related to both symmetries. However, it is possible to have a perfect vacuum condition after the simultaneous application of both symmetries, namely, the symmetries under changes in the price plus the symmetries under changes in volatility. This result comes out to be similar to the one analyzed in Equation (63) if we have $\rho = 0$ or $\zeta = 0$, as it was explained previously.

7.1.5. Weak-Field Condition $\phi_x^n << \phi_x$ and $\phi_y^m << \phi_y$; for Any Value of n, m

For this regime, Equation (56) gives the approximation

$$-\frac{e^y}{2}n(n-1)\phi_y^2 - \rho\zeta e^{y(\alpha-1/2)}nm\phi_x\phi_y - \zeta^2 e^{2y(\alpha-1)}m(m-1)\phi_x^2 \approx 0, \tag{65}$$

which for $n = m = 1$ would give again $\phi_{x\ vac}\phi_{y\ vac} \approx 0$ as before. We can certainly solve this previous equation as a quadratic equation in either ϕ_x or ϕ_y. If we decide to solve for ϕ_y, we get

$$\phi_{y\ vac} \approx \frac{\rho\zeta e^{y(\alpha-3/2)}m\phi_{x\ vac}}{(1-n)}\left(1 \pm \sqrt{1 - \frac{2(m-1)(n-1)}{\rho^2 nm}}\right). \tag{66}$$

Note the coupling between $\phi_{y\ vac}$ and $\phi_{x\ vac}$, such that the triviality of one field implies the triviality of the other. However, it is still possible to have a trivial value for $\phi_{y\ vac}$ and still a non-trivial one for $\phi_{x\ vac}$ if $\zeta = 0$ or if $m = 1$. The coupling between fields means that the broken symmetry, with respect to the changes in the prices, is connected to the broken symmetry with respect to changes in volatility. This also means that the random fluctuations of the prices are tied to the random fluctuations of the volatility. This naturally is connected to the flow of information inside the system.

7.1.6. Weak and Strong-Field Approximation: $\phi_x^n >> \phi_x$ and $\phi_y^m << \phi_y$

For this case, Equation (56) becomes

$$\left(\lambda e^{-y} + \mu - \frac{\zeta^2}{2}e^{2y(\alpha-1)}\right)\phi_x^2\phi_y - \zeta^2 e^{2y(\alpha-1)}(m-1)\phi_x^2 \approx 0, \tag{67}$$

which gives a trivial solution for $\phi_{x\ vac} \approx 0$ and the following result for $\phi_{y\ vac}$

$$\phi_{y\ vac} \approx \frac{\zeta^2 e^{2y(\alpha-1)}(1-m)}{\lambda e^{-y} + \mu - \frac{\zeta^2}{2}e^{2y(\alpha-1)}}. \tag{68}$$

Then, for this case, the symmetry under changes in the prices is not broken, but the one related to the changes in volatility is spontaneously broken, except when $m = 1$ or $\zeta = 0$. Then, in the most general situations, the random fluctuations cannot affect the prices of the market under this approximation. Still, the random fluctuations generate changes in the information content of the system because the location of the equilibrium condition changes due to the random fluctuations on the volatility. Once again, a value of $\zeta = 0$ decouples the Gaussian noise from the evolution in time of the volatility. Note that the approximation (67) requires the additional condition $\phi_x >> \phi_y$, under the assumption that all the coefficients in the Hamiltonian (56) are of the same order of magnitude. Of course, other different regimes where different terms are relevant (or irrelevant) can be analyzed.

7.1.7. Weak and Strong-Field Approximation: $\phi_x^n << \phi_x$ and $\phi_y^m >> \phi_y$

For this case, Equation (56) is approximated to

$$\frac{e^y}{2}(n-1)\phi_y^2 + \left(r - \frac{e^y}{2}\right)\phi_x\phi_y^2 \approx 0. \tag{69}$$

This equation gives a trivial result for $\phi_{y\ vac} \approx 0$ and a non-trivial one for $\phi_{x\ vac}$ as

$$\phi_{x\ vac} \approx \frac{(1-n)\frac{e^y}{2}}{r - \frac{e^y}{2}}, \tag{70}$$

which vanishes when $n = 1$ or when $y \to -\infty$ ($\sigma^2 \to 0$). Then for this case, in general, the symmetry under changes of the volatility is a perfect symmetry, meanwhile, the symmetry under changes of the prices is spontaneously broken. Then the random fluctuations only affect the the information contained on the prices in the market in these special circumstances. Interestingly, when $y \to \infty$, the previous expression becomes

$$\phi_{x\ vac} \approx n - 1, \tag{71}$$

which is independent from any parameter and only depends on the order of the term in the series expansion under analysis. Note that the approximation (69) requires the additional complementary condition $\phi_y >> \phi_x$. Of course, other regimes different to the ones analyzed here are possible. However, the selected regimes are enough for illustrating the power of the proposed formulation.

8. Conclusions

In this paper, we have analyzed the degeneracy of the martingales state on regimes which were not considered before in previous papers. We have also connected the concept of spontaneous symmetry breaking in Quantum Finance with the flow of information through the market. The analysis has been done for both the BS equation as well as for the case where the volatility is stochastic, namely, for the MG equation. For the BS equation as well as for the MG one, the symmetries under changes of the prices of the options cannot annihilate the vacuum (taken as the martingale state). These symmetries are then spontaneously broken in both cases. This means that random fluctuations in the market generate changes in the information of the system, forcing it to move toward another equilibrium condition. We found the conditions where the vacuum, in these two situations (MG and BS equations), becomes unique. These come out to be the same conditions for which there is no flow of information through the boundaries of the system (Market), or equivalently, the same conditions correspond to the cases where the Hamiltonian becomes Hermitian. Hermitian Hamiltonians, in general, preserve the information of the system. Under the ordinary definition of martingale, the connection between flow of information and spontaneous symmetry breaking is exact. We could then extend the martingale condition for including the symmetries under changes of the stochastic volatility, in a similar fashion to what is done in [8]. We then defined the momentum for the volatility as the generator of these transformations. We could also find the conditions under which the martingale state (vacuum) becomes unique for different regimes and for the different series expansion terms. A unique vacuum or martingale state means that the random fluctuations cannot change the information of the system under such special situations. In [8], the authors only focused on the strong field approximation. Here, however, we could complete the scenario by analyzing the weak-field approximation for the BS as well as for the MG case. In the weak-field approximation, the kinetic terms are the most relevant. Since the kinetic terms, expressed as a function of a quantum field, behave as potential terms, their existence then affects the vacuum location and behavior. For the case of the MG equation, we have to define two different quantum fields, one corresponding to the prices in the market and the other related to the volatility. In this paper, we analyzed the regimes where both fields in the MG equation are either strong or weak. Additionally, we analyzed the situations where one field is strong and the other is weak. In these cases, it came out that one of the symmetries involved were perfect but the other one was spontaneously broken in general, except for some specific combination of parameters. Perfect symmetries mean conservation of the information under random fluctuations. This is a very important statement for the stock market analysis. The use of symmetry arguments in order to analyze the flow of information in the stock market and equilibrium conditions proposed here can be complemented, with the formalism employing fractional functionals [42] in order to have a deeper visualizations about the flow of information in the market. This part will be developed in future papers. It is also important to remark that since the flow of information in the stock market is related to whether or not the symmetries of the system are spontaneously broken, then there should be a natural connection between the concept of spontaneous symmetry breaking and decision theory. This immediately brings us the chance to analyze possible connections between symmetry patterns and game theory. Previous relations between quantum formulations and game theory were proposed in [43]. Finally, some previous studies suggest that modified diffusion equations might emerge from the BS equation [44]. This point deserves attention in connection with the flow of information in the stock market and its relation with the symmetries of the system.

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