

Review

# The Role of the Volatility in the Option Market

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**Abstract:** We review some general aspects about the Black–Scholes equation, which is used for predicting the fair price of an option inside the stock market. Our analysis includes the symmetry properties of the equation and its solutions. We use the Hamiltonian formulation for this purpose. Taking into account that the volatility inside the Black–Scholes equation is a parameter, we then introduce the Merton–Garman equation, where the volatility is stochastic, and then it can be perceived as a field. We then show how the Black–Scholes equation and the Merton–Garman one are locally equivalent by imposing a gauge symmetry under changes in the prices over the Black–Scholes equation. This demonstrates that the stochastic volatility emerges naturally from symmetry arguments. Finally, we analyze the role of the volatility on the decisions taken by the holders of the options when they use the solution of the Black–Scholes equation as a tool for making investment decisions.

**Keywords:** stock market; decision theory; volatility; Black–Scholes equation; Merton–Garman equation; option price

**MSC:** 81S99; 91B80; 91G15



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## 1. Introduction

A financial market is defined as the place where two or more individuals buy and sell financial assets and make general transactions. The stock market is a well known financial market, and it consists of trading shares in the stocks of some company or organization [1]. Inside the stock market, some derivative instruments are also traded. One important derivative instrument traded in the market is the option. Options were traded for the first time in 1973 [2]. Interestingly, the same year, Black and Scholes introduced their famous Black–Scholes (BS) formula [3]. This formula gives the fair price of an option as a function of the stock price, time to maturity, strike price, interest rate and volatility [2]. Since the formula is dynamical, then the time as a variable also appears inside the solution. The BS formula is by itself a differential equation containing two free-parameters, namely, the volatility and the interest rate [4]. The brilliance of the BS formulation is in the fact that it proposed a way to cancel random fluctuations by mixing up inside a single portfolio the prices of a security (stock price) with the prices of a derivative instrument. This cancellation of the random fluctuations is what makes the BS equation a predictive one [3,4]. Different methods for analyzing the option prices have been developed previously. Among them, we have the fractional-order option pricing [5] and the Laplace homotopy analysis method (LHAM) using the Caputo–Fabrizio (CF) fractional derivative operator [6]. Additionally, some ideas trying to modify the BS equation and compare it with variance models were proposed in [7]. In general, the BS equation, as a differential equation, can be expressed as a Hamiltonian (eigenvalue) equation after some variable transformations [4]. This formulation gives us significant advantages with respect to the standard differential equation formulation. Among the numerous advantages of the Hamiltonian formulation, we have the following: (1). The problem becomes an eigenvalue problem such as in quantum mechanics [8]. (2). We can identify the free-parameters that control the dynamic of the system,

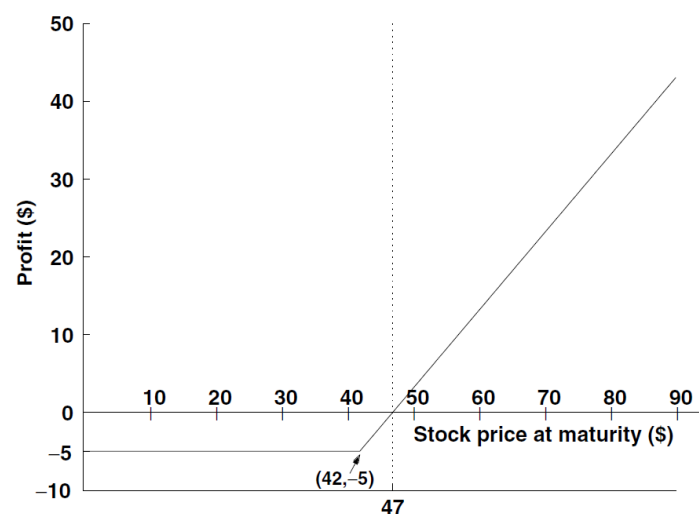
which, in the case of the BS equation, are the volatility and interest rate. (3). We can analyze the symmetries of the system and then identify symmetry breaking patterns [9–11]. (4). We can identify the operators generating transformations in certain quantities, such as the prices of the stocks, for example, or even the volatility if we consider the Merton–Garman equation [12,13]. (5). We can simply use analogies with quantum mechanics (QM) in order to develop several interesting new tools inside the financial market [4,8–11,14–17]. This does not mean that the financial market is quantum; this rather means that we can use certain analogies, with the advantage of knowing in advance the form of the solutions for certain equations. However, we must be careful at the moment of working over these analogies because there are certain differences between the Hamiltonians appearing in ordinary quantum mechanics [8] and the financial Hamiltonians, such as in the BS case. One strong difference is that in ordinary QM, the Hamiltonians are normally Hermitian [8], the Hamiltonians in finance, including the BS Hamiltonian, are generically non-Hermitian [4,9–11]. Then we have loss of unitarity in the stock market, or equivalently, the information is not preserved inside the stock market [10,17]. Although, in QM, non-unitarity is considered to be a serious problem [8], inside the stock market, the loss of information is, in some sense, expected. However, even in ordinary physics, there are cases where non-Hermiticity is valid, as far as the  $PT$  symmetry is satisfied [18–24]. However, this is not the case inside the stock market. In this paper, we focus our analysis on the role of the volatility in the option market. For this purpose, we show the solutions for the BS equation and we explain how the investors use them to decide whether they should buy an option inside the market or not. From this perspective, we explain how the investors estimate the volatility value and how they compare it with the value obtained from the implied volatility [2]. Finally, given the importance of the volatility inside the stock market, in this paper, we revise the seminal result demonstrating that the BS equation is locally equivalent to the Merton–Garman (MG) equation [11]. This result is a natural consequence of imposing the local symmetry condition under changes in the prices of the stocks over the BS Hamiltonian. In this way, the stochastic volatility emerges as a field necessary for satisfying a local symmetry over the BS equation. The local equivalence of the BS and the MG equations is a very important result because it means that the volatility as a variable emerges from a series expansion around the BS solution. The paper is organized as follows: In Section 2, we revise the concept of the option market. We then explain the difference between call option and put option. In Section 3, we explain some generalities about the BS equation. We then derive its Hamiltonian form and then use the corresponding Hamiltonian for calculating certain quantities. In Section 4, we analyze the Merton–Garman equation (MG), which is an extension of the BS case in order to include the volatility field as a stochastic variable. We then explain how to obtain the corresponding Hamiltonian, and then we use it for calculating certain quantities. In Section 5, we analyze the interesting mechanism of spontaneous symmetry breaking inside the scenario of quantum finance by using the BS equation as an example where the mechanism appears. In Section 6, we demonstrate the local equivalence between the BS equation and the MG one. In Section 7, we explain the Higgs mechanism describing the dynamical origin of the volatility. In Section 8, we analyze the solutions of the BS equation, which is used by the investors in general for making decisions in the market. Finally, in Section 9, we conclude.

## 2. The Option Market

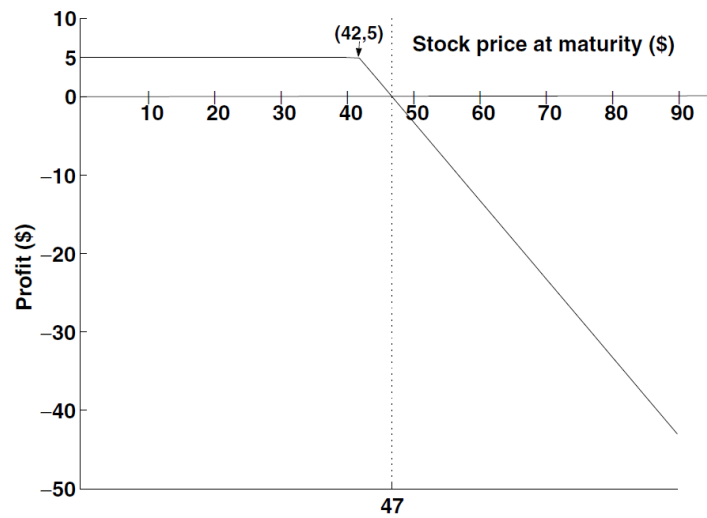
The options were created and traded for the first time in 1973. Interestingly, during the same year, the famous Black–Scholes (BS) formula was introduced [2]. This equation has been used since then by investors and academics for determining the fair price of an option. Before exploring the details of the BS equation, we have to understand how the option market operates. The option market usually has a holder of the option and the writer of the same option [4]. There are two types of options: (1). The call option, where the holder has the right to buy the shares of a stock at some predetermined price. (2). The put option, where the holder has the right to sell the shares of some stock at some predetermined price.

The stock market charts, such as those appearing in Yahoo Finance [25], give us the price  $S(t)$  of different stocks corresponding to different companies. An option is nothing else than a contract signed between the holder (buyer of the option) and a writer (seller of the option). For the call options, the holder expects the prices of the stock to increase (bullish), and the writer expects the same prices to decrease (bearish). This is the case because for a call option, the holder has the right to buy a stock in future at the same price as the stock is priced today (at the moment of signing the contract). Then the holder is interested in a tendency to increase the value of the option that he holds.

The holder and the writer of an option, at the most fundamental level, are playing a zero-sum game, as it can be perceived when we compare the Figures 1 and 2. Indeed, Figure 1 is the mirror image of Figure 2, showing in this way that the profits of the holder are the losses of the writer and vice versa. In this particular example, the holder buys a call option at 5 USD per share. This option gives him the right to buy some specific stock of some company at the strike price of 42 USD. For the holder to exercise the option, he/she needs it to have at least a stock price larger than 47 USD per share. This is the case because the holder bought the option at 5 USD, starting then with a negative balance of 5 USD per share, as is illustrated in Figure 1. He/she has this negative balance when he/she does not exercise the option, which is the case when the stock price is inferior to 47 USD per share. If by the expiration date of the Option, the price is larger than 47 USD per share, then the holder earns some positive profits after exercising the option. It is at this point where the writer starts to lose profits, as can be seen from Figure 2, which analyzes the profits of the writer for the same option and stock price. Note that the writer earns positive profits if the option is not exercised by the holder. In summary, the holder earns money when he exercises the option after verifying that the price of the same option is larger than 47 USD (for closing prices of the stock between 47 USD and 42 USD, the European option could be also exercised in order to minimize the losses suffered by the holder). What he earns (the holder) is the difference between the price of the stock minus what he/she paid for the option, multiplied by the number of shares involved in the process. On the other hand, the writer makes money when the holder does not exercise the option after verifying that the stock price closes at a value inferior to 47 USD (more strictly for stock values inferior to 42 USD because for stock prices between 42 USD and 47 USD, the holder could still exercise the European call option for minimizing his/her losses). Here, we want to remark that while the maximum possible profit for the writer is 5 USD per share (see Figure 2), the maximum profit for the holder is just limited by the price of the stock involved in the process.

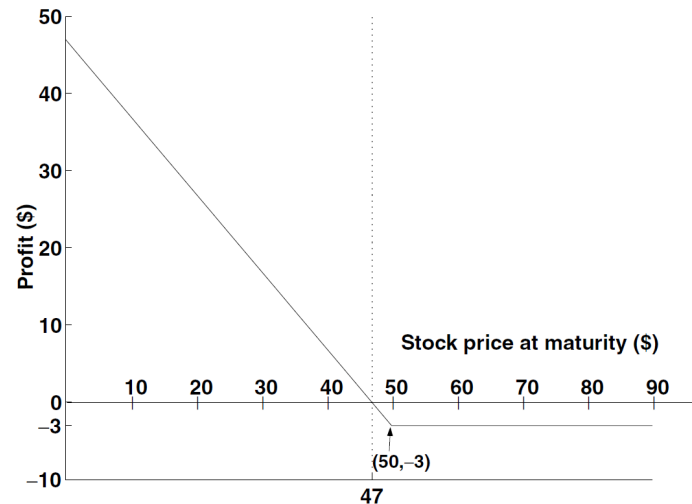


**Figure 1.** Example of the possible profits received by the holder of a European call option. The option price is 5 USD, and the strike price is 42 USD. This is an example of a zero-sum game. Figure taken from [2].



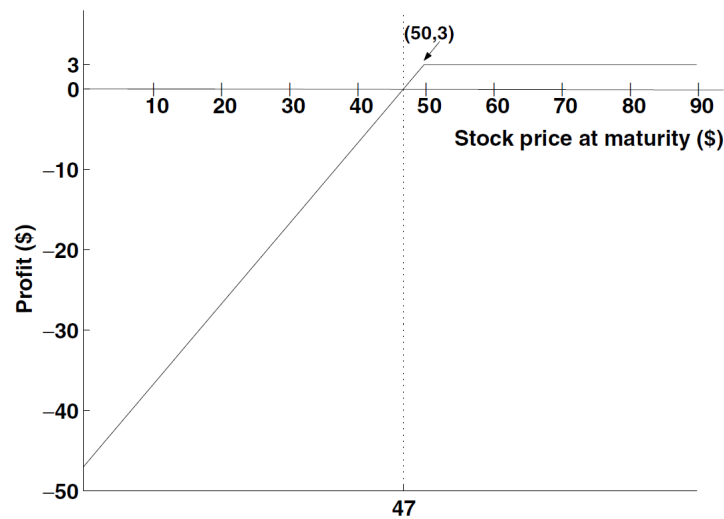
**Figure 2.** Example of the possible profits received by the writer of a European call option. The option price is 5 USD and the strike price is 42 USD. This is an example of a zero-sum game. Figure taken from [2].

We can now analyze the situation with the put option. A put option is one where the holder, once he/she buys it, he/she has the right to sell a stock at some specific pre-arranged price. This means that the holder of the option pays to the writer of the option some amount in order to obtain the right to sell a stock at a pre-specified price, which is usually the price of the stock on the day of buying the option. In this case, the holder of the option wants the price of the stock to fall (bearish), while the writer of the option wishes for the prices to increase (bullish). Figure 3 illustrates an example for the possible profits earned by the holder of a European call option. We can notice that the lower the price of the stock is, the larger the earnings of the holder.



**Figure 3.** Example of the possible profits received by the holder of a European put option. The option price is 3 USD and the strike price is 50 USD. This is an example of a zero-sum game. Figure taken from [2].

In this particular example, the cost of the option is 3 USD. Then naturally, the holder loses 3 USD per share and then the writer of the same option earns 3 USD per share, as can be seen from Figure 4.



**Figure 4.** Example of the possible profits received by the writer of a European put option. The option price is 3 USD and the strike price is 50 USD. This is an example of a zero-sum game. Figure taken from [2].

Figures 3 and 4 are mirror images of each other. This is a typical relation for the pay-offs between two players in a zero-sum game. Inside the European call option, for this example, the holder receives positive pay-offs when the price of the stock is inferior to 47 USD. This is the case because the holder has the right to sell the stock at 50 USD, no matter what happens with the prices of the stock. Since he/she has to recover the investment of 3 USD, then he/she only exercises the option if the price of the involved stock falls below 47 USD (The holder could also decide to exercise the European put option for prices of the stock between 47 USD and 50 USD, if by the time of closing, he/she wishes to minimize his/her losses). For prices of the stock larger than this value (47 USD), the holder will still possibly exercise the option if the price of the stock is inferior to 50 USD, if only to minimize losses. However, if the prices of the stock increase over 50 USD, exercising the option brings losses larger than 3 USD per share to the holder. Then the option in this case is never exercised and the writer earns 3 USD per share, while the holder loses the same amount. For this particular example, the highest pay-off for the holder is 47 USD, while the writer also loses the same amount in this situation. Here again, the highest possible pay-off received by the writer is just the price of the option per share, and in the same situation, the holder would lose the same amount.

### 3. The Black–Scholes Equation

In this paper, we consider the stock price as  $S(t)$ , which is normally taken as a random stochastic variable evolving in agreement to a stochastic differential equation, given by

$$\frac{dS(t)}{dt} = \phi S(t) + \sigma S(t)R(t). \tag{1}$$

Here,  $\phi$  is the expected return of the security,  $R(t)$  is the Gaussian white noise with zero mean and  $\sigma$  is the volatility. If we exclude the volatility, then we can guarantee the evolution of the price of the stock with certainty [4]. In this way, by imposing  $\sigma = 0$ , we obtain a simple solution for Equation (1) as

$$S(t) = e^{\phi t} S(0). \tag{2}$$

Additionally, the possibility of arbitrage is excluded if we can make a perfect hedged portfolio. The key ingredient inside the analysis of Black and Scholes was to eliminate any

possibility of uncertainty, such that we can analyze the evolution of the price free of any white noise [4]. We can then consider the following portfolio:

$$\Pi = C - \frac{\partial C}{\partial S} S. \tag{3}$$

This is a portfolio where an investor holds the option and then short sells the amount  $\frac{\partial C}{\partial S}$  for security  $S$ . The price of the option  $C$  and its derivative  $\frac{\partial C}{\partial S}$  are naturally correlated, and then, any random fluctuation can be eliminated through this correlation. By using the Ito calculus (stochastic calculus) [4], it is possible to demonstrate that

$$\frac{d\Pi}{dt} = \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}. \tag{4}$$

Here, the change in the value of  $\Pi$  does not have uncertainty associated with it [4]. Any random fluctuation has vanished because the random term has disappeared due to the choice of portfolio. Since, in this case, we have a risk-free rate of return (no arbitrage) [26,27], then the following equation is satisfied

$$\frac{d\Pi}{dt} = r\Pi. \tag{5}$$

If we use the results (3) and (4) together with the previous equation, then we obtain

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC. \tag{6}$$

This is the Black–Scholes equation and its details and assumptions can be found in [4,12,13].

### 3.1. Black–Scholes Hamiltonian Formulation

Here, we explain how Equation (6) can be expressed as an eigenvalue problem after a change in variable. The resulting equation is the Schrödinger equation with a non-Hermitian Hamiltonian [8]. For Equation (6), consider the change of variable  $S = e^x$ , where  $-\infty < x < \infty$ . In this way, the BS equation becomes

$$\frac{\partial C}{\partial t} = \hat{H}_{BS} C, \tag{7}$$

where we have defined the operator

$$\hat{H}_{BS} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{1}{2} \sigma^2 - r\right) \frac{\partial}{\partial x} + r, \tag{8}$$

as the BS Hamiltonian [4,9–11]. Note that the resulting Hamiltonian is non-Hermitian since  $\hat{H} \neq \hat{H}^\dagger$  [14,15]. If we compare with the standard Schrödinger equation, defined as [8]

$$\hat{H}\psi(x, t) = E\psi(x, t) = \frac{\partial}{\partial t}\psi(x, t), \tag{9}$$

then it is evident that the BS Hamiltonian is defined in Equation (8), and that this Hamiltonian can help us to understand the evolution of the prices of the options. The result (8) has important consequences when we analyze the probability flow inside the scenario of the BS equation [10]. In addition, note that since the spot interest rate  $r$  is constant, then the potential term is simply a constant term. This means that the vacuum condition is trivial for this case. Under the BS Hamiltonian, the evolution in time of the option is non-unitary in general (in addition, the Hamiltonian non-necessarily obeys the  $PT$  symmetry). This means that the probability is not necessarily preserved in time inside the BS model, although it is certainly well-defined, and its total value is equal to one, if the normalization factor is updated from time to time. In general, as was mentioned in the introduction, there are some cases in ordinary quantum mechanics, as well as in quantum field theory, where it is interesting to explore non-Hermitian Hamiltonians (Lagrangians) [18,20,28]. Based on

the previous explanations, we cannot expect the financial market to obey unitarity. This is precisely the point that we want to analyze in deep detail in this paper.

### 3.2. The Evolution of Probability in the Black–Scholes Equation

The solutions for the BS equation are well-known [2,4], and we explore them subsequently. Before analyzing the solutions, we study the evolution of the prices of the options, taking advantage of the Hamiltonian formulation. We can then find the kernel for the evolution of the prices in the stock market. The kernel naturally depends on the Hamiltonian under analysis. To understand this issue, we can solve the Schrödinger equation defined in Equation (9), obtaining then

$$\psi(t, x) = e^{iEt} \psi(0, x). \tag{10}$$

Here  $E$  is the eigenvalue for the Hamiltonian  $\hat{H}$ . The same result can be expressed in the bra–ket notation as

$$|\psi, t\rangle = e^{Et} |\psi, 0\rangle. \tag{11}$$

Considering the final condition at  $t = T$ , we have

$$|\psi, T\rangle = e^{ET} |\psi, 0\rangle = g, \tag{12}$$

where  $g$  can be interpreted as the pay-off function if we assume  $|\psi\rangle = |C\rangle$ . Under this equality, the initial option price is then defined as

$$|C, 0\rangle = e^{-ET} g, \tag{13}$$

and then

$$|C, t\rangle = e^{-E\tau} g, \tag{14}$$

with the time defined as  $\tau = T - t$  and then running backwards. This means that in this formulation, the initial condition  $t = 0$ , is equivalent to  $\tau = T$ , which, inside the stock market scenario, corresponds to the maturity time for the option. The solution (14) represents an exponential decaying behavior. If we project the result (14) over  $\langle x|$ , then we obtain the functional behavior, as follows:

$$C(x, t) = \langle x|C, t\rangle = \langle x|e^{-\tau H}|g\rangle = \int_{-\infty}^{\infty} dx' \langle x|e^{-\tau H}|x'\rangle g(x'). \tag{15}$$

Note that in the last step, we have used the completeness relation  $\int_{-\infty}^{\infty} dx'|x'\rangle \langle x'| = \hat{I}$  for the basis of the stock prices, represented by the variable  $x$ . Then, we define the pricing kernel as

$$p(x, \tau, ; x') = \langle x|e^{-\tau \hat{H}}|x'\rangle. \tag{16}$$

### 3.3. Black–Scholes Pricing Kernel

By introducing the Hamiltonian (8) inside Equation (16), we should be able to find the explicit form of the pricing kernel for the BS case. However, we have to develop a few definitions initially, as follows [4]:

$$\delta(x - x') = \langle x|x'\rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x-x')} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x|p\rangle \langle p|x'\rangle. \tag{17}$$

The completeness relation,

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} |p\rangle \langle p| = \hat{I}, \tag{18}$$

is satisfied and this means that we are taking the basis of the prices as a complete basis. With the previous results, we can define the product as

$$\langle x|p\rangle = e^{ipx}, \quad \langle p|x\rangle = e^{-ipx}. \tag{19}$$

The BS kernel can be obtained by using Equation (16) with the BS Hamiltonian  $\hat{H} = \hat{H}_{BS}$ . We can re-express Equation (16) as

$$p_{BS}(x, \tau, ; x') = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x|e^{-\tau\hat{H}_{BS}}|p\rangle \langle p|x'\rangle. \tag{20}$$

If we want to find the explicit result, we need to evaluate the following matrix element explicitly:

$$\langle x|\hat{H}_{BS}|p\rangle = E_{BS}\langle x|p\rangle = E_{BS}e^{ipx}. \tag{21}$$

Here,  $E_{BS}$  is the eigenvalue of the Hamiltonian operator corresponding to the state  $|p\rangle$ . We can find the explicit value of  $E_{BS}$  if we evaluate the following matrix element:

$$\langle p|\hat{H}_{BS}|x\rangle = \langle x|\hat{H}_{BS}^+|p\rangle^* = \left(\frac{1}{2}\sigma^2 p^2 + i\left(\frac{1}{2}\sigma^2 - r\right)p + r\right)e^{-ipx}. \tag{22}$$

Here, we are using  $\hat{p}|p\rangle = p|p\rangle$  because in this case, the operator  $\hat{H}_{BS}$  is acting over the momentum space. In position space, on the other hand, the momentum operator is a derivative operator. If we use the result (22), then the kernel defined by Equation (20) becomes

$$p_{BS}(x, \tau, ; x') = e^{-r\tau} \frac{1}{\sqrt{2\pi\tau\sigma^2}} e^{-\frac{1}{2\tau\sigma^2}\left(x-x'+\tau\left(r-\frac{\sigma^2}{2}\right)\right)^2}. \tag{23}$$

Here again,  $\tau = T - t$ . The kernel (23) allows us to analyze the probability flow in the market, and then we can analyze the non-conservation of probability in this case. First, we can evaluate the time-derivative of the kernel as [10]

$$\frac{\partial p_{BS}}{\partial t} = \frac{\partial p_{BS}}{\partial \tau} \frac{\partial \tau}{\partial t}. \tag{24}$$

Since  $\partial\tau/\partial t = -1$ , then we obtain

$$\frac{\partial p_{BS}}{\partial t} = -\frac{\partial p_{BS}}{\partial \tau} \neq 0, \tag{25}$$

in general. The previous derivative is equal to zero for some relations between the variables and between parameters. In fact, we obtain

$$r = -\frac{\sigma^2}{2}, \quad \text{if} \quad \frac{\partial p_{BS}}{\partial t} = 0, \quad \tau \rightarrow \infty. \tag{26}$$

We also have the additional condition

$$\tau = \frac{(x-x')^2}{\sigma^2}, \quad \text{if} \quad \frac{\partial p_{BS}}{\partial t} = 0, \quad \tau \rightarrow 0, \quad (\text{arbitrary } r). \tag{27}$$

It is interesting to notice that in Equation (26), the risk-free interest rate takes negative values under the imposed limit. In fact,  $r$  can be either positive or negative [29]. However, in order to satisfy the demands of the BS equation, we impose the condition  $r \geq 0$ , avoiding negative risk-free interest rates. From Equation (26), we can see that an infinite maturity time is necessary, under the condition  $r = -\sigma^2/2$ , in order to preserve the probability in the market, something which can only be approximated in some circumstances. In Equation (27), the condition for preserving the probability is the logical fact that for small scales of maturity time, we can know with certainty the market behavior. This, in addition, implies that  $x \rightarrow x'$ , which means that the stock prices do not change so much under the conditions proposed in Equation (27). On the other hand, since Equation (26) allows the



possibility of having negative values for  $r$ , there must be a limit value for the time  $\tau$  where  $r = 0$ . This important limit can be found if we impose  $r \rightarrow 0$  in the time-derivative of the kernel (23). Assuming again the conservation of probability ( $\frac{\partial p_{BS}}{\partial t} = 0$ ), we obtain the polynomial

$$\tau^2 + \frac{4}{\sigma^2}\tau - 4\frac{(x - x')^2}{\sigma^4} = 0. \tag{28}$$

If we solve for  $\tau$ , then we obtain

$$\tau_{r=0} = -\frac{2}{\sigma^2} \left( 1 \pm \sqrt{1 + (x - x')^2} \right). \tag{29}$$

Here, the lowest sign is the correct one in order to keep  $\tau$  positive. The limit value  $\tau_{min}^{r=0} = 0$  ( $t = T$ , final value considering that the evolution of the system is analyzed from the final condition toward the initial one) again corresponds to the condition  $x \rightarrow x'$ , which is consistent with the fact that  $x'$  corresponds to the final price. Note that when  $x \gg x'$ , namely, when  $(x - x') \rightarrow \infty$  such that the price at maturity is much larger than the initial price of the stock, then we have the result

$$\tau_{max}^{r=0} \approx \frac{2|x - x'|}{\sigma^2}. \tag{30}$$

Another important limit to be considered is  $r = \frac{\sigma^2}{2}$  because it represents the limit when the BS system preserves the information, because in this limit, the BS Hamiltonian becomes Hermitian. For this special limit, the condition  $\frac{\partial p_{BS}}{\partial t} = 0$  gives

$$\tau = -\frac{1}{4r} \left( 1 \pm \sqrt{1 + \frac{8r(x - x')^2}{\sigma^2}} \right). \tag{31}$$

Since we know again that the condition  $\tau \geq 0$  must be satisfied, we select the lowest sign in Equation (31). Note that for the minimal value of  $\tau_{min}^{r=\sigma^2/2} = 0$ , we obtain

$$\frac{r(x - x')^2}{\sigma^2} = 0. \tag{32}$$

This limit corresponds to the case where the stock is close to maturity. If, additionally, we assume  $r \neq 0$  and  $\sigma \neq \infty$  (finite volatility), then we obtain

$$x \rightarrow x', \quad \text{when } \tau \rightarrow \tau_{min}^{r=\sigma^2/2} = 0, \tag{33}$$

which again shows the consistency of the present formulation. Here again, we can predict the largest value for  $\tau$  (the largest maturity time) by considering the limit  $|x - x'| \rightarrow \infty$ . For this case, we obtain

$$\tau_{max}^{r=\sigma^2/2} \approx \sqrt{\frac{1}{2r}} \left( \left| \frac{x - x'}{\sigma} \right| \right) = \frac{|x - x'|}{\sigma^2}, \tag{34}$$

where we have used  $r = \sigma^2/2$  for this special case. The limit  $r = \sigma^2/2$  is extremely important because it corresponds to the case where the BS Hamiltonian becomes Hermitian, and for this case, the information (unitarity) should be preserved in principle. The currents of probability associated with the BS equation were analyzed in [10].

### 3.4. The Influence of the Non-Conservation of the Probability in the Prices of the Stocks under the Black-Scholes Formulation

The prices in the stock are represented by the variable  $x$ . This variable then affects the behavior of the option prices, here represented by  $C(x, t)$ . The way this function evolves is contained in the kernel defined in Equation (23). It is evident that in the stock market, the probability is not conserved. For this case, the conservation of probability only appears with

respect to the analytically extended time  $\mu$  for the case  $r = \frac{\sigma^2}{2}$  [10]. This special condition appears when the non-Hermitian portion of the Hamiltonian  $\hat{H}_{BS}$  vanishes. Then, under the Hermiticity condition, the kernel (23) becomes

$$p_{BS}^H = e^{-r\tau} \frac{1}{\sqrt{2\pi\tau\sigma^2}} e^{-\frac{1}{2\tau\sigma^2}(x-x')^2}. \tag{35}$$

This result corresponds to a standard Gaussian distribution with center  $x'$  and with time-dependent height. If we compare with Equation (23), we notice that the effect of the non-Hermitian portion of the BS Hamiltonian is to create a shift in the location of the center of the Gaussian. In such a case, the center of the Gaussian is also time-dependent. Under analytical extension, we can perceive from Equation (35) that the Kernel corresponding to the Hermitian portion of the BS Hamiltonian is just a plane-wave with a time-dependent amplitude (decreasing amplitude actually). Under the same analytical extension, the full kernel (23) has some additional time-dependence on the amplitude of the function. Then, we can conclude that in fact, the loss of information due to the presence of the non-Hermitian contributions in the BS Hamiltonian affects the location of the most probable price of the stock to be observed, as well as the total effect of the volatility

#### 4. The Merton–Garman Equation

In this section, we consider the MG equation. Inside the MG scenario, the volatility and the price of the stock are both stochastic. In this situation, the market is incomplete because the volatility cannot be traded in the market [4]. Although several stochastic processes have been considered for modeling the case with stochastic volatility [30–36], here, we consider the generic case, defined by the set of equations [4]

$$\begin{aligned} \frac{dS}{dt} &= \phi S dt + S\sqrt{V}R_1, \\ \frac{dV}{dt} &= \lambda + \mu V + \zeta V^\alpha R_2. \end{aligned} \tag{36}$$

Here, the volatility is defined by the stochastic variable  $V = \sigma^2$ , and  $S$  is the stochastic variable representing the price of the stock. Additionally,  $\phi$ ,  $\lambda$ ,  $\mu$  and  $\zeta$  are free-parameters of the system [26].  $R_1$  is the Gaussian noise for the price of the stock, and  $R_2$  is the same but for the case of volatility.  $R_1$  and  $R_2$  are correlated in the following way:

$$\langle R_1(t')R_1(t) \rangle = \langle R_2(t')R_2(t) \rangle = \delta(t - t') = \frac{1}{\rho} \langle R_1(t)R_2(t') \rangle. \tag{37}$$

Here,  $-1 \leq \rho \leq 1$ , and the bra–kets  $\langle AB \rangle$  correspond to the correlation between  $A$  and  $B$ . If we consider a function  $f$ , depending on  $S(t)$ , explicitly on the time, as well as on the white noises  $R_1$  and  $R_2$ , and with the help of the Ito calculus, it is possible to derive the total derivative in time of this function as

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \phi S \frac{\partial f}{\partial S} + (\lambda + \mu V) \frac{\partial f}{\partial V} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 f}{\partial S^2} + \rho V^{1/2+\alpha} \zeta \frac{\partial^2 f}{\partial S \partial V} + \frac{\zeta^2 V^{2\alpha}}{2} \frac{\partial^2 f}{\partial V^2} + \sigma S \frac{\partial f}{\partial S} R_1 \\ &\quad + \zeta V^\alpha \frac{\partial f}{\partial V} R_2. \end{aligned} \tag{38}$$

The compact expression for this equation is

$$\frac{df}{dt} = \Theta + \Xi R_1 + \psi R_2, \tag{39}$$

with the corresponding definitions

$$\Theta = \frac{\partial f}{\partial t} + \phi S \frac{\partial f}{\partial S} + (\lambda + \mu V) \frac{\partial f}{\partial V} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 f}{\partial S^2} + \rho V^{1/2+\alpha} \zeta \frac{\partial^2 f}{\partial S \partial V} + \frac{\zeta^2 V^{2\alpha}}{2} \frac{\partial^2 f}{\partial V^2}, \quad \Xi = \sigma S \frac{\partial f}{\partial S}, \quad \psi = \zeta V^\alpha \frac{\partial f}{\partial V}, \quad (40)$$

This is the same notation used in [3].

#### 4.1. Derivation of the Merton–Garman Equation

If we consider two different options defined as  $C_1$  and  $C_2$  on the same underlying security (stock price) with strike prices and maturities given by  $K_1, K_2, T_1$  and  $T_2$ , respectively, then it is always possible to create a portfolio,

$$\Pi = C_1 + \Gamma_1 C_2 + \Gamma_2 S, \quad (41)$$

that combines both Options. If we consider the result (39), then the total derivative for the folio with respect to time is

$$\frac{d\Pi}{dt} = \Theta_1 + \Gamma_1 \Theta_2 + \Gamma_2 \phi S + (\Xi_1 + \Gamma_1 \Xi_2 + \Gamma_2 \sigma S) R_1 + (\psi_1 + \Gamma_1 \psi_2) R_2. \quad (42)$$

This result is obtained after defining  $f(t) = C_1$  or  $f(t) = C_2$  in Equation (39). It has been demonstrated that even in this particular case, where we have stochastic volatility, it is still possible to generate a hedged folio, and then at the end, we arrive again to the condition  $\frac{d\Pi}{dt} = r\Pi$ , which is the typical one satisfying the requirements of complete markets. This important result is obtained after removing the white noises  $R_1$  and  $R_2$ . This is implemented after imposing the following conditions:

$$\begin{aligned} \psi_1 + \Gamma_1 \psi_2 &= 0, \\ \Xi_1 + \Gamma_1 \Xi_2 + \Gamma_2 \sigma S &= 0, \end{aligned} \quad (43)$$

and then solving for  $\Gamma_1$  and  $\Gamma_2$ . The solution for  $\Pi$  is non-trivial for this case, and then it requires the definition of the parameter

$$\begin{aligned} &\beta(S, V, t, r) = \\ &\frac{1}{\partial C_1 / \partial V} \left( \frac{\partial C_1}{\partial t} + (\lambda + \mu V) \frac{\partial C_1}{\partial S} + \frac{VS^2}{2} \frac{\partial^2 C_1}{\partial S^2} + \rho V^{1/2+\alpha} \zeta \frac{\partial^2 C_1}{\partial S \partial V} + \frac{\zeta^2 V^{2\alpha}}{2} \frac{\partial^2 C_1}{\partial V^2} - rC_1 \right) \\ &= \frac{1}{\partial C_2 / \partial V} \left( \frac{\partial C_2}{\partial t} + (\lambda + \mu V) \frac{\partial C_2}{\partial S} + \frac{VS^2}{2} \frac{\partial^2 C_2}{\partial S^2} + \rho V^{1/2+\alpha} \zeta \frac{\partial^2 C_2}{\partial S \partial V} + \frac{\zeta^2 V^{2\alpha}}{2} \frac{\partial^2 C_2}{\partial V^2} - rC_2 \right). \end{aligned} \quad (44)$$

This parameter does not appear for the case of the Black–Scholes equation for example. This is the case because  $\beta$  in the MG equation is the market price volatility risk. This can be understood after observing the behavior of the parameter  $\beta$ . In this way, the higher the value of  $\beta$  is, the lower the intention of the investors to risk. This is a natural consequence of the fact that in the MG equation, the volatility is a stochastic variable. Since the volatility is not traded in the market, then it is not possible to make a direct hedging process over this quantity [4]. Then, when we have stochastic volatility, the expectations of the investors are important and they are reflected on the parameter  $\beta$ . Previously, it was demonstrated in [37] that the value of  $\beta$  in agreement with Equation (44) is always a non-vanishing result. In general, it is always assumed that the risk of the market (in price) has been included inside the MG equation. The MG equation is then obtained by rewriting the Equation (44) in the form

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + (\lambda + \mu V) \frac{\partial C}{\partial V} + \frac{1}{2} VS^2 \frac{\partial^2 C}{\partial S^2} + \rho \zeta V^{1/2+\alpha} S \frac{\partial^2 C}{\partial S \partial V} + \zeta^2 V^{2\alpha} \frac{\partial^2 C}{\partial V^2} = rC. \quad (45)$$

In this equation, the effects related to the intentions of the investors, which were originally stored in  $\beta$ , now appear contained inside the parameter  $\lambda$ . In other words, we have shifted the parameter  $\lambda \rightarrow \lambda - \beta$  in Equation (45). In what follows, we transform the MG equation

to its Hamiltonian form, which is very convenient for analyzing some interesting properties of the MG equation.

#### 4.2. Hamiltonian Form of the Merton–Garman Equation

The MG equation can be also formulated as a Hamiltonian (eigenvalue) equation after performing the appropriate redefinition of variables. The changes in variables in this case are

$$\begin{aligned} S &= e^x, & -\infty < x < \infty, \\ \sigma^2 &= V = e^y, & -\infty < y < \infty, \end{aligned} \tag{46}$$

and then, the MG Equation (45) becomes [4,16,38]

$$\begin{aligned} \frac{\partial C}{\partial t} + \left(r - \frac{e^y}{2}\right) \frac{\partial C}{\partial x} + \left(\lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)}\right) \frac{\partial C}{\partial y} + \frac{e^y}{2} \frac{\partial^2 C}{\partial x^2} + \rho \zeta e^{y(\alpha-1/2)} \frac{\partial^2 C}{\partial x \partial y} \\ + \zeta^2 e^{2y(\alpha-1)} \frac{\partial^2 C}{\partial y^2} = rC. \end{aligned} \tag{47}$$

We can interpret this equation here as an eigenvalue equation. In such a case, we have

$$\frac{\partial C}{\partial t} = \hat{H}_{MG} C, \tag{48}$$

with the MG Hamiltonian defined as

$$\begin{aligned} \hat{H}_{MG} = -\frac{e^y}{2} \frac{\partial^2}{\partial x^2} - \left(r - \frac{e^y}{2}\right) \frac{\partial}{\partial x} - \left(\lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)}\right) \frac{\partial}{\partial y} - \rho \zeta e^{y(\alpha-1/2)} \frac{\partial^2}{\partial x \partial y} \\ - \zeta^2 e^{2y(\alpha-1)} \frac{\partial^2}{\partial y^2} + r. \end{aligned} \tag{49}$$

Some exact solutions for the MG equation have been found for the special cases where  $\alpha = 1$ , as was implemented in [16] by using path-integral techniques. The same equation has also been solved in [30–36] for the case  $\alpha = 1/2$  by using standard techniques of differential equations. The equation has two degrees of freedom, and they are transmitted to its Hamiltonian. The MG Hamiltonian, as is the case of the BS Hamiltonian, is non-Hermitian, and the non-conservation of probability is expected.

#### 4.3. The Merton–Garman Kernel

The flow of probability for the MG case can be analyzed when we again calculate its kernel in the same way it was calculated for the BS case [4]. The kernel describes the evolution of the system between two well-defined prices, which are represented by a variable  $x$ , as follows

$$p_{MG}(x, \tau | x') = \langle x | e^{-\tau \hat{H}_{MG}} | x' \rangle, \quad \tau = T - t, \tag{50}$$

with  $T$  representing the maturity price. By following the same techniques explained in [4], we find

$$\langle \mathbf{p} | \hat{H}_{MG} | \mathbf{x} \rangle = \left( \frac{e^y}{2} p_x^2 + i \left( \frac{e^y}{2} - r \right) p_x - i \left( \lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)} \right) p_y + \rho \zeta e^{y(\alpha-1/2)} p_x p_y - \frac{\zeta^2 e^{2y(\alpha-1)}}{2} p_y^2 + r \right) e^{-i \mathbf{p} \cdot \mathbf{x}}.$$

Here,  $\mathbf{p} \cdot \mathbf{x} = p_x x + p_y y$ , with  $x$  connected with the prices and  $y$  related to the volatility in the market. In Equation (49), we can see that a portion of the MG Hamiltonian looks like

the BS one with  $\sigma^2 = e^y$ . Then, one portion of the kernel looks similar to the BS kernel. We can now define the MG kernel in agreement with Equation (50) as

$$p_{MG}(x, \tau; x') = \int_{-\infty}^{\infty} \frac{dp_x dp_y}{4\pi^2} \langle x | e^{-\tau \hat{H}_{MG}} | p \rangle \langle p | x' \rangle. \tag{51}$$

Introducing Equation (51) inside the previous result, we obtain

$$p_{MG}(x, \tau; x') = e^{-r\tau} \int_{-\infty}^{\infty} \frac{dp_x dp_y}{4\pi^2} e^{-\frac{1}{2}\tau e^y p_x^2} \times e^{ip_x(x-x'+\tau(r-\frac{e^y}{2}))} e^{\frac{1}{2}\tau \zeta^2 e^{2y(\alpha-1)} p_y^2} e^{ip_y \tau (\lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)})} e^{-\tau \rho \zeta e^{y(\alpha-1/2)} p_x p_y}. \tag{52}$$

This kernel contains all the time-dependence of the prices of the stock plus the fluctuations in volatility (variations in time of volatility).

#### 4.4. The Flow of Probability in the MG Case

The evolution of the prices and all the dynamics in general is concentrated in the kernel. Then, it is useful to evaluate the possible variations in the kernel with respect to time. Here again, since  $\tau = T - t$ , then we have  $\partial p_{MG} / \partial t = -\partial p_{MG} / \partial \tau$ . In [17], it was demonstrated that for the case of the stock market, the following relation is valid:

$$\frac{\partial \rho}{\partial t} = 2C \frac{\partial C}{\partial t}. \tag{53}$$

This equation expresses the evolution of the probability for some price of the option to occur. This probability is defined as  $\rho = |C(x, t)|^2$ . Here,  $C(x, t)$  is a functional distribution of the prices of an option. If we use the Schrödinger equation defined in (48), we obtain

$$\begin{aligned} \frac{\partial \rho}{\partial t} = 2r|C|^2 - e^y C \frac{\partial^2 C}{\partial x^2} - 2\left(r - \frac{e^y}{2}\right) C \frac{\partial C}{\partial x} - 2\rho \zeta e^{y(\alpha-1/2)} C \frac{\partial^2 C}{\partial x \partial y} - 2\zeta^2 e^{2y(\alpha-1)} C \frac{\partial^2 C}{\partial y^2} \\ - 2\left(\lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)}\right) C \frac{\partial C}{\partial y}. \end{aligned} \tag{54}$$

If we integrate over the whole range of stock prices, as well as over the whole range of possible volatility values (expressed through the variable  $y$ ), then we obtain the following expression:

$$\begin{aligned} \frac{\partial P_{MG}}{\partial t} = 2rP_{MG} + \int_{-\infty}^{\infty} dy \left(\frac{e^y}{2} - r\right) (C(y)^2)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx \left(\lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)}\right) (C(x)^2)_{-\infty}^{\infty} \\ - \int_{-\infty}^{\infty} dx dy C \left( e^y \frac{\partial^2 C}{\partial x^2} + 2\rho \zeta e^{y(\alpha-1/2)} \frac{\partial^2 C}{\partial x \partial y} + 2\zeta^2 e^{2y(\alpha-1)} \frac{\partial^2 C}{\partial y^2} \right). \end{aligned} \tag{55}$$

This equation suggests that in general, the probability is not conserved in the stock market. Even under asymptotically well-behaved solutions for the prices of the option, we would obtain approximately the solution

$$\frac{\partial P_{MG}}{\partial t} \approx 2rP_{MG}, \tag{56}$$

which is equal to the Black–Scholes case under the same conditions [17]. This part is the generic non-conservation of probability because it appears in any Financial equation, no matter how the volatility behaves. Equation (56) has the solution

$$P_{MG} \approx e^{2rt}, \tag{57}$$

up to some constant, normalizing the result. The information is, however, not preserved here because the normalization factor changes in time. This means that the total probability

at one instant is normalized to be equal to one; however, at a subsequent time, the same probability is not normalized anymore under the same normalization factor. Then, a new normalization factor would be required for keeping consistency. This problem is called loss of unitarity in quantum mechanics [8], and in general, it represents a big problem in fundamental science. However, at the stock market level, it is expected that the information is not preserved, and we should not worry so much about the loss of unitarity. The unitarity of any system is restored when the Hermiticity of the Hamiltonian is restored [8]. Then, if we analyze the Hamiltonian of the MG equation defined in Equation (49), we understand the the non-Hermitian contributions disappear when the conditions

$$r = \frac{e^y}{2}, \quad \frac{\zeta^2}{2} e^{2y(\alpha-1)} - \lambda e^{-y} - \mu = 0 \tag{58}$$

are satisfied. These conditions then guarantee the conservation of probability but not with respect to the ordinary time but with respect to the imaginary time-coordinate, here defined as  $t \rightarrow -i\omega$ , as it can be easily proven [10,17]. The result (58) can be summarized in the following expression:

$$\lambda + 2\mu r - 2^{2(\alpha-1)} \zeta^2 r^{2\alpha-1} = 0. \tag{59}$$

For the special case where  $\alpha = 1$ , we can then solve the result for  $r$ , as follows:

$$r = \frac{\lambda}{\zeta^2 - 2\mu}. \tag{60}$$

Additionally, for the special case with  $\alpha = 1/2$ , we obtain

$$r = \frac{\zeta^2/2 - \lambda}{2\mu}. \tag{61}$$

The results (60) and (61) correspond to the special values that the risk-free interest rate has to take for some special values of  $\alpha$  such that the market can still preserve unitarity or, in other words, preserve the probability. For more general values of  $\alpha$ , we only have to solve the Equation (59) with the corresponding value taken by  $\alpha$ .

### 5. Spontaneous Symmetry Breaking from the Black–Scholes Hamiltonian

Spontaneous symmetry breaking is a phenomenon taken from physics [39,40]. It consists of the fact that under certain circumstances, the ground state of a system can violate certain symmetries that are still satisfied by the Hamiltonian. The typical example is the Mexican hat case as it appears in the Figure 5. For illustrating this example, we can consider the following Hamiltonian corresponding to the linear  $\sigma$ -model [39,41]:

$$\hat{H} = \int d^3x \left( \frac{1}{2} (\partial_\mu \phi^i)^2 - \frac{1}{2} \mu^2 (\phi^i)^2 + \frac{\lambda}{4!} (\phi^i)^4 \right). \tag{62}$$

From this Hamiltonian, the potential term corresponds to the expression ignoring the kinetic terms, as follows:

$$V(\phi) = -\frac{1}{2} \mu^2 (\phi^i)^2 + \frac{\lambda}{4!} (\phi^i)^4. \tag{63}$$

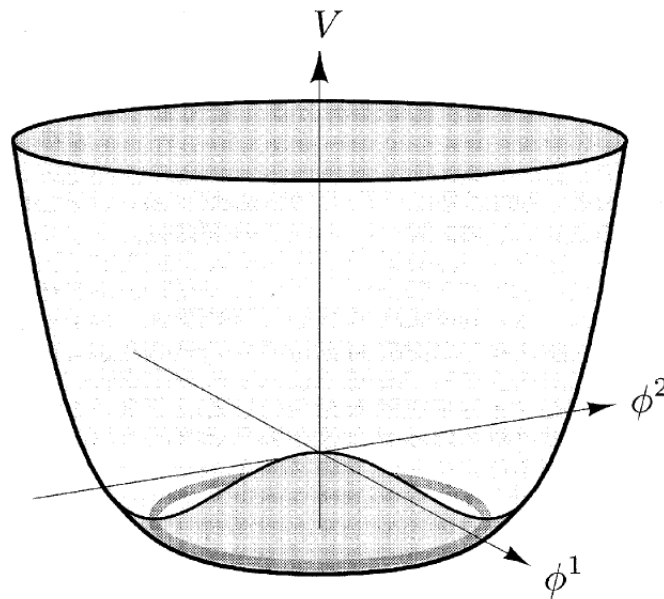
The Hamiltonian (62), and then the potential (63), are invariant under the symmetry transformation  $\phi^i \rightarrow R^{ij} \phi^j$ . The ground state for the potential (63) is

$$(\phi_0^i)^2 = \frac{\mu^2}{\lambda}. \tag{64}$$

The field  $\phi^i$  in this example has  $N$ -components. Then, the vacuum state condition (64) does not specify the direction taken by the ground state. It only specifies its magnitude. For example, we could select a ground state consistent with (64), as follows:

$$\phi_0^i = (0, 0, 0, 0, \dots, v), \quad v = \frac{\mu}{\sqrt{\lambda}}. \tag{65}$$

Then, the ground state selects a preferred direction, and it does not respect the symmetry of the original Hamiltonian (62). From the ground state (65), we can see that the system could perfectly select  $N$ -different components for the ground state. All these ground states have the same energy, and we call them degenerate ground states [39,40,42–47]. An additional remark is that the ground state (64) would be trivial if  $\mu^2 < 0$ . However, when  $\mu^2 > 0$ , the non-trivial result (64) remains. This means that whether the symmetry is spontaneously broken or not depends on the values taken by the free-parameters of the system. Further details about the mechanism of spontaneous symmetry breaking can be found in [39]. In this paper, we review how to apply this principle when we deal with the financial Hamiltonians already mentioned. We start with the BS Hamiltonian because all the financial Hamiltonians are locally equivalent to the BS Hamiltonian [11]. Subsequently, we demonstrate that the BS Hamiltonian and the MG one are equivalent locally after imposing local symmetry conditions over the BS equation.



**Figure 5.** Mexican hat potential, typical for potentials of the form defined in Equation (63). Figure taken from [41].

5.1. The Martingale Condition as a Vacuum State

The fundamental theorem of finance establishes the existence of a martingale measure as far as the market is complete and the condition of no-arbitrage holds [4]. The martingale condition is a risk-neutral measure, and it suggests that the conditional probability of the discounted value of an equity at time  $t \neq 0$  is equal to its present value at  $t = 0$  [1,48–50]. This can be expressed mathematically as

$$S(0) = E[e^{-\int_0^t r(t')dt'} S(t) | S(0)]. \tag{66}$$

In this equation,  $S(t)$  is the stock price, and  $r$  is the short-term risk-free interest rate. In the same equation,  $t'$  is a dummy variable representing the time, integrated from zero up to a certain value as the limits of the integral suggest. In ordinary situations, the martingale condition is unique. Here, we can formulate this condition in its Hamiltonian form. For

this purpose, we can consider the standard change in variable  $S = e^x$ , which allows us to express the martingale condition as

$$S(x) = \int_{-\infty}^{\infty} dx' \langle x | e^{-(t_*-t)\hat{H}} | x' \rangle S(x'). \tag{67}$$

If we use the complete base condition  $\int_{-\infty}^{\infty} dx |x\rangle \langle x| = \hat{I}$ , with  $\hat{I}$  being the identity, and in addition  $S(x) = \langle x | S \rangle$ , then we obtain

$$|S \rangle = e^{-(t_*-t)\hat{H}} |S \rangle. \tag{68}$$

This is equivalent to

$$\hat{H} |S \rangle = 0. \tag{69}$$

This is the ground state of the financial system, and it coincides with the martingale condition. The martingale condition, expressed in the form (69), can be also verified by applying the explicit form of the financial Hamiltonian over the state  $|S \rangle$  [4].

### 5.2. Broken Symmetries in the Financial Equations

The symmetries of a system are defined by the operators (generators of symmetries) that commute with the Hamiltonian of the system in the following way:  $[\hat{H}, \hat{A}] = 0$ . If the same symmetry is not respected by the ground state of the system, then we have the additional condition  $\hat{A} |S \rangle \neq 0$ . Under these circumstances, we say that the symmetry of the system is spontaneously broken [9,11]. If we translate this language to quantum finance, this means that the martingale state does not obey the same symmetries of the financial Hamiltonian. In order to illustrate this phenomena in finance, we can start with the BS Hamiltonian [4]. The Schrödinger equation is

$$\hat{H}_{BS} C(x, t) = EC(x, t), \tag{70}$$

with the BS Hamiltonian defined as in Equation (8). Here,  $\langle x | \hat{p} | C \rangle = \partial C(x, t) / \partial x$ . It can be proven that if we replace the option  $C(x, t)$  in Equation (70) by the stock price  $S(x, t) = e^x$ , then we obtain the vacuum condition (69). Note, however, that although the operator defined as  $\hat{p} C(x, t) = \partial C(x, t) / \partial x$  is a conserved quantity (symmetry of the Hamiltonian), the same operator does not annihilate the ground state (martingale state), as can be seen from the following expression:

$$\hat{p} |S \rangle \neq 0. \tag{71}$$

This means that the ground state does not respect the symmetry under changes in prices, even if the BS Hamiltonian does. Then, this symmetry is spontaneously broken and the martingale state is represented by a degenerate vacuum. This is in some sense expected because this means that any value taken by the stock price  $S(x, t) = e^x$  represents a potential martingale state, as far as the market is complete and there is no chance of arbitrage. Then, the market can reach the equilibrium for any value of  $S(x, t)$ . Equivalently, for any ket  $|S \rangle$ , we have potentially a martingale condition if the market is in equilibrium. If  $\hat{p}$  is extended to the complex plane such that it becomes  $\hat{p} \rightarrow i\hat{p}$ , then spontaneously broken symmetry would correspond to rotations on the complex plane. These rotations are of the form  $U = e^{-i\hat{p}\alpha}$ .

### 5.3. Spontaneous Symmetry Breaking: Symmetries under Changes of Prices

We can use the result (71) to derive some expressions by defining the martingale state as  $\langle x | S \rangle = e^x = \sum_n \phi^n$ , such that  $\langle x | S \rangle = S(x, t)$ . The effect of the broken generator  $\hat{p}$  over the field  $\phi$  defined in this way is to map it toward another field  $\bar{\phi}$  with different components. In the same way, applying  $\hat{p}$  over  $\bar{\phi}$  appropriately maps it back to the original



field  $\phi$ . Without loss of generality, we can select  $\bar{\phi}$  to satisfy  $\langle S|\bar{\phi}|S \rangle = 0$ . With these definitions, we have

$$\langle S|[\hat{p}, \bar{\phi}]|S \rangle = \int_{-\infty}^{\infty} dx S(x, t) \frac{\partial}{\partial x} \bar{\phi}(x, t) \neq 0, \tag{72}$$

which is obtained from

$$\int_{-\infty}^{\infty} dx \langle S|\hat{p}|x \rangle \langle x|\bar{\phi}|S \rangle - \int_{-\infty}^{\infty} dx \langle S|\bar{\phi}|x \rangle \langle x|\hat{p}|S \rangle, \tag{73}$$

and taking into account the completeness relation  $\int_{-\infty}^{\infty} dx |x \rangle \langle x| = \hat{I}$ , together with the standard expression  $\langle x|\hat{p}|S \rangle = \partial S/\partial x$ . In addition, we can define  $\langle x|\bar{\phi}|S \rangle = \bar{\phi}(x, t)$ . These results are general and they do not depend on how we define the field  $\bar{\phi}$ . With these arguments, we can demonstrate that  $\langle S|\phi|S \rangle \neq 0$  because  $\langle S|\sum_n \phi^n|S \rangle = e^x$ , which never vanishes unless  $x \rightarrow -\infty$ . Note, however, that here we have fixed  $\bar{\phi}$  as a quantum field satisfying  $\langle S|\bar{\phi}|S \rangle = 0$ , which precisely corresponds to  $x \rightarrow -\infty$ . This is the Nambu–Goldstone field as it is defined equivalently in [39]. In addition, it is well-known that the effect of  $\hat{p}$  is to move the system from one vacuum to another one. Then for example, the system can be translated from the vacuum defined by  $\bar{\phi}$  toward a new vacuum defined by  $\phi$ . Following the standard terminology, we can define the order parameter as

$$\langle S|[\hat{p}, \bar{\phi}]|S \rangle = \langle S|\phi|S \rangle \neq 0, \tag{74}$$

which is valid if

$$\hat{p}\bar{\phi}(x) = \frac{\partial \bar{\phi}(x)}{\partial x} = \phi(x), \tag{75}$$

taking into account that

$$\langle S|\phi|S \rangle = \int_{-\infty}^{\infty} dx \langle S|x \rangle \langle x|\phi|S \rangle = \int_{-\infty}^{\infty} dx S(x, t) \phi(x), \tag{76}$$

in general. The consistency of this expression with the results (72) and (74) suggests the validity of the condition (75). Then, what the broken symmetry generator  $\hat{p}$  does is to map one vacuum toward another one. This is typical from spontaneous symmetry breaking. We can have a better understanding of this phenomena if we define the potential of the Hamiltonian (8), as follows:

$$\hat{V} = \left( \frac{1}{2}\sigma^2 - r \right) \hat{p} + r. \tag{77}$$

If we analyze the minimum for this potential, then the state  $C(x, t)$  approaches to  $S(x, t) = e^x$  near the ground state (martingale state). Considering the expansion  $S(x, t) = e^x = \sum_{n=0}^{\infty} (x)^n/n! = \sum_{n=0}^{\infty} \phi^n(x, t)$ , we can define the potential (77) as a function of  $S(x, t)$  taken as a field as

$$V(S) = \langle x|\hat{V}|S \rangle = \sum_n \left( \frac{1}{2}\sigma^2 - r \right) n\phi^{n-1}(x) + r\phi^n(x). \tag{78}$$

Here, we have used the definition of  $\phi^n(x, t)$  and the fact that  $\partial \phi^n/\partial x = n\phi^{n-1}$ . This result is obtained if we consider  $\partial S/\partial x = e^x = \sum_{n=0}^{\infty} (x)^{n-1}/(n-1)! = \sum_{n=0}^{\infty} n\phi^{n-1}(x, t)$ . This means that  $\sum_n \partial \phi^n(x, t)/\partial x = \sum_n \phi^n(x, t) = \sum_n n\phi^{n-1}(x, t)$ . At the moment of comparing terms in the potential, ideally we perform it order by order. However, at any order in the expansion, the relative exponent for the terms in the expansion do not change. Then, without loss of generality, we can select any order in the expansion of  $n$  to perform the

corresponding calculations. Then, we can focus on the second order terms in the expansion ( $n = 2$ ), as follows:

$$V(S) = \langle x | \hat{V} | S \rangle \approx 2 \left( \frac{1}{2} \sigma^2 - r \right) \phi(x) + r \phi(x)^2. \tag{79}$$

The relative exponent between the derivative and the non-derivative term is the most important point to compare. The relation between exponents does not change, no matter which order in the expansion we decide to compute. In this way, for simplicity, we can consider Equation (79) as the appropriate potential for studying the vacuum conditions of the system. The minimum of the potential can be expressed as a function of the free-parameters of the system, namely,  $\sigma$  and  $r$ . The potential defined in Equation (79) contains a term that comes from the derivative-term in Equation (77) ( $\hat{p}S(x, t) = \partial S(x, t) / \partial x$ ). This term is the source of non-Hermiticity, and it is responsible for the non-conservation of information. In other words, this term is related to the flow of information through the boundaries of the system [17]. The minimum for the potential (79) occurs when  $\frac{\partial V(S)}{\partial \phi} = 0$ . Solving for  $\phi(x)$ , we find

$$\phi_{vac} = 1 - \frac{\sigma^2}{2r}. \tag{80}$$

This vacuum condition clearly depends on the free-parameters of the system. The free-parameters are, naturally, the interest rate and volatility. If  $r = \sigma^2/2$ , then the vacuum is trivial since  $\phi_{vac} = 0$ . This naturally corresponds to a trivial value of the security  $S(x, t)$ . The same relation guarantees the no-flow of information through the boundaries of the system (Hermiticity). Here, we take  $\sigma^2 \leq 2r$  conventionally to avoid the theory being unstable. Then, the largest possible value for the volatility is constrained by the value of the interest rate  $r$ . In addition, when  $r \gg \sigma^2$ , the vacuum converges to a constant value. The relation (104) is non-unique because its value depends on the values taken by the free-parameters of the system.

### 6. Local Equivalence between the Black–Scholes and the Merton–Garman Equation

In several cases, it comes out that two apparently different theories are equivalent locally. The same occurs with certain groups of equations belonging to the corresponding theories. One important example happens when we go from special (SR) to general relativity (GR), for example [51]. In such a case, both theories are locally equivalent, and their differences appear when we compare global aspects of both theories. Such global aspects are mathematically equivalent to the introduction of gauge fields [52]. In GR, the Christoffel connections are equivalent to these gauge-fields, as can be seen from several different theories of gravity [53]. The presence of gauge fields modifies the definition of derivative. Then for example, when we are working in SR, the covariant derivative for a vector  $V^\nu$  is defined as

$$\partial_\mu A^\nu = \frac{\partial}{\partial x^\mu} A^\nu. \tag{81}$$

However, when we go to GR, the definition of derivative is modified to

$$\nabla_\mu A^\nu = \frac{\partial}{\partial x^\mu} A^\nu + \Gamma_{\mu\gamma}^\nu A^\gamma. \tag{82}$$

Globally, Equations (81) and (82) are different. However, locally they are equivalent because in such a case, the factors  $\Gamma_{\mu\gamma}^\nu$  vanish. In GR this is called the equivalence principle and this is the reason why, in a free-falling frame of reference, an observer does not feel gravity, for example. Interestingly, we can translate similar arguments to other theories. We call these sets of theories gauge theories, and the symmetries involved are called gauge symmetries. In general terms, a gauge theory introduces physical (gauge) fields that restore the local symmetry of the Hamiltonian (Lagrangian) of the theory under analysis. The new physical

(gauge) field normally appears to modify the definition of derivative inside the Hamiltonian. In this section, we show that the MG equation emerges naturally from the BS equation after imposing gauge symmetries over the BS equation [11].

*The Merton–Garman Equation Emerging from the Black–Scholes Equation*

We study how the BS Hamiltonian changes when a local transformation under some shift on the prices emerges. We can consider a local transformation under the changes in prices as  $U = e^{\omega\theta(x)}$ . Here,  $\theta(x)$  is a variable depending on  $x$ , which also depends on the price of the stock  $S(t)$ , as Equation (46) suggests. If the operator  $U$  were a symmetry of the system, then it would satisfy the condition  $[\hat{H}_{BS}, U] = 0$ . However, it is possible to demonstrate that this is not the case here after using the Hamiltonian given in Equation (8), together with the definition of the local changes in price. In this way, we obtain

$$[\hat{H}_{BS}, U] \neq 0. \tag{83}$$

If we want to obtain an exact symmetry under local changes in the prices ( $U = e^{\omega\theta(x)}$ ), then we need to add additional terms inside the Hamiltonian (8). This can be perceived after applying the transformation  $U = e^{\omega\theta(x)}$  over the BS Hamiltonian, as follows:

$$\hat{H}_{BS} \rightarrow \hat{H}_{BS} + \frac{\sigma^2\omega(1+\omega)}{2} \left(\frac{\partial\theta(x)}{\partial x}\right)^2 + \sigma^2\omega \left(\frac{\partial\theta(x)}{\partial x}\right) \frac{\partial}{\partial x} + \omega \left(\frac{1}{2}\sigma^2 - r\right) \frac{\partial\theta(x)}{\partial x}. \tag{84}$$

If we want to keep the invariance under the action of  $U$ , then we have to add some additional terms to the Hamiltonian such that the symmetry is restored. These additional terms are not added arbitrarily. The best way to add them is by modifying the definition of derivative as it was exemplified in Equation (82) for the case of general relativity. Here, for the BS equation, we redefine the derivative to become a covariant derivative: defined as

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \hat{p}_y. \tag{85}$$

Looking into the analogy with the gauge theories, we interpret  $\hat{p}_y$  as the gauge field that restores the symmetry of the system. It comes out that this gauge field is related to the stochastic volatility of the system. More exactly,  $\hat{p}_y$  can be interpreted as the momentum associated with the volatility. After replacing the ordinary derivative with the covariant derivative in Equation (8), we obtain

$$\hat{H}_{BS} \rightarrow \hat{H} = \frac{\sigma^2}{2} (-\hat{p}_x - \hat{p}_y)(\hat{p}_x + \hat{p}_y) + \left(\frac{1}{2}\sigma^2 - r\right) (\hat{p}_x + \hat{p}_y) + r. \tag{86}$$

The minus sign difference in the products appearing in the first term is a natural consequence of the fact that the momentum associated with the changes in prices, as well as the momentum associated with the changes in the stochastic volatility are both non-Hermitian quantities, satisfying then the conditions

$$\hat{p}_x^+ = \frac{\partial}{\partial x}^+ = -\frac{\partial}{\partial x}, \quad \hat{p}_y^+ = \frac{\partial}{\partial y}^+ = -\frac{\partial}{\partial y}. \tag{87}$$

Here, the index + means Hermitian conjugate operation. Equation (86) can be expressed as

$$\hat{H} = -\frac{\sigma^2}{2} \hat{p}_x^2 + \left(\frac{1}{2}\sigma^2 - r\right) \hat{p}_x - \frac{\sigma^2}{2} \hat{p}_y^2 - \sigma^2 \hat{p}_x \hat{p}_y + \left(\frac{1}{2}\sigma^2 - r\right) \hat{p}_y + r. \tag{88}$$

The gauge invariance under a general transformation of the form  $U = e^{\omega\theta(x,y)}$  for the new financial Hamiltonian defined in Equation (88) is satisfied if the following conditions are accomplished

$$\begin{aligned} \left(\frac{\partial\theta}{\partial x}\right)^2 &= \frac{\omega}{1+\omega}\left(\frac{\partial\theta}{\partial y}\right)^2, \\ \left(\frac{\partial\theta}{\partial x}\right)\hat{p}_x &= \left(\frac{\partial\theta}{\partial y}\right)\hat{p}_y, \\ \frac{\partial\theta}{\partial x} + \frac{\partial\theta}{\partial y} - 4\frac{\partial^2\theta}{\partial x\partial y} &= \frac{2r}{\sigma^2}\left(\frac{\partial\theta}{\partial x} + \frac{\partial\theta}{\partial y}\right). \end{aligned} \tag{89}$$

These conditions are obtained after checking the invariance of Equation (88). Then, the changes due to local transformations of the new terms appearing in Equation (88) exactly balances the additional terms appearing in Equation (84). Interestingly, when  $\sigma^2 = 2r$ , then  $\frac{\partial^2\theta}{\partial x\partial y} = 0$ . This is also the same Hermiticity condition for the BS Hamiltonian, which would guarantee conservation of information. The symmetry is then completely restored after including the volatility as a gauge field inside the BS Hamiltonian, and this is independent from the free-parameters of the theory. The only consideration we need to make in order to restore the local symmetry of the Hamiltonian is to select the appropriate values for the function  $\theta(x, y)$ . In summary, the gauge invariance of the Hamiltonian (88), under local changes of the stock prices, is fully satisfied. Interestingly, it comes out that the Hamiltonian defined in Equation (88) is just the Merton–Garman Hamiltonian if we redefine the parameters appropriately, as follows:

$$\begin{aligned} \zeta^2 &= e^{-2y(\alpha-\frac{3}{2})}, \\ \rho\zeta &= e^{-y(\alpha-\frac{3}{2})}, \\ r &= \lambda e^{-y} + \mu. \end{aligned} \tag{90}$$

These expressions guarantee the equivalence of the Hamiltonian in Equation (88) and the MG Hamiltonian [4]. Interestingly, the relations in Equation (90) give us the conditions  $\rho = \pm 1$ , which correspond to the extreme conditions for the parameter  $\rho$ . We have to remember that in the standard MG equation, the parameter  $\rho$  satisfies the following condition [4]:

$$-1 \leq \rho \leq 1. \tag{91}$$

Then, as far as we define the covariant derivative in the form (85), the gauge invariance of the BS Hamiltonian suggests that the MG parameter  $\rho$  can only take the extreme values  $\pm 1$  from the condition (91). If this is true, then the white noises related to the time evolution of the stock price and volatility satisfy the following conditions:

$$\langle R_1(t)R_1(t') \rangle = \langle R_2(t)R_2(t') \rangle = \pm \langle R_1(t)R_2(t') \rangle, \tag{92}$$

under the mentioned circumstances where the gauge invariance connects the BS and the MG equations [4,10]. There is an interesting connection between the interest rate  $r$  and the two volatility coefficients  $\lambda$  and  $\mu$  inside Equation (90). This means that the local equivalence between the BS equation and the MG equation gives a natural connection between the interest rate and volatility of the market.

### 7. The Higgs Mechanism in Quantum Finance: The Dynamical Origin of the Volatility

It should not be a surprise that the stochastic volatility in the market, inside the scenario of gauge theories, behaves as a massive term inside the financial Hamiltonian. Here, we remark once again that the Hamiltonian obtained in Equation (88) is the Merton–

Garman Hamiltonian as long as the conditions (90) are satisfied. Before, in [9,10], the ground (martingale) state for the MG Hamiltonian was analyzed. It was found to be the most general martingale condition for the MG equation, which is also consistent with Equation (88) and is given by

$$\hat{H}_{MG}e^{x+y} = \hat{H}_{MG}S(x, y, t) = 0. \tag{93}$$

We can consider this as the martingale condition for the Hamiltonian (88). In [9], the martingale condition (93) is a real vacuum condition, only when the constraint,

$$\lambda + e^y \left( \mu + \frac{\zeta^2}{2} e^{2y(\alpha-1)} + \rho\zeta e^{y(\alpha-1/2)} \right) = 0, \tag{94}$$

is satisfied. With the constraints defined in Equation (90), Equation (94) becomes equivalent to the condition

$$e^{2y} + \mu e^y + \lambda = 0. \tag{95}$$

If we solve this equation, we obtain

$$e^y = -\frac{\mu}{2} \left( 1 \mp \sqrt{1 - \frac{4\lambda}{\mu}} \right). \tag{96}$$

This equation suggests an interesting relation between the stochastic volatility and the parameters  $\lambda$  and  $\mu$ , a consequence of the martingale condition (93). This can be seen explicitly if we introduce the last relation in Equation (90) inside Equation (95). In such a case, we obtain  $e^y = -r$ , which connects the interest rate  $r$  with the volatility (defined through  $y$ ). Here, however, a negative interest rate emerges in order to have the general market equilibrium defined in Equation (93). However, we must remark that the martingale state suggested in Equation (93) is not the only possible definition of a martingale state inside the MG scenario. Still, Equation (93) is the only definition for martingale, considering simultaneously the prices of the stock plus the volatility. Although the martingale condition has some arbitrariness, it is still universal to define it as the ground state of the MG Hamiltonian defined in Equation (88). The potential to minimize is [9,10].

$$\hat{V} = \left( \frac{1}{2}\sigma^2 - r \right) \hat{p}_x - \sigma^2 \hat{p}_x \hat{p}_y + \left( \frac{1}{2}\sigma^2 - r \right) \hat{p}_y + r. \tag{97}$$

Here we consider the terms linear in  $\hat{p}_x$  and  $\hat{p}_y$  as potential terms. We can compare Equation (97) with the standard MG Hamiltonian defined in [9], and repeated here as

$$\hat{V}(x, y) = -\left( r - \frac{e^y}{2} \right) \hat{p}_x - \left( \lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)} \right) \hat{p}_y + r. \tag{98}$$

It is a trivial task to show that the Equations (97) and (98) are the same under the conditions (90). In the neighborhood of the minimal defined by the condition (93), we can consider that  $\langle x, y | S \rangle = S(x, y, t) = e^{x+y} = \sum_{n=0}^{\infty} (x+y)^n / n! = \sum_{n=0}^{\infty} \phi_x^n \phi_y^n$ . Additionally, we know that  $\partial S(x, y, t) / \partial x = \partial S(x, y, t) / \partial y = \sum_n (x+y)^{n-1} / (n-1)! = e^{x+y} = \sum_n n \phi_x^{n-1} \phi_y^n = \sum_n n \phi_x^n \phi_y^{n-1}$ . If, here again, we limit the series expansion to second order, then the potential term to analyze is

$$\langle x, y | \hat{V}(x, y) | S \rangle = V(S) = -2 \left( r - \frac{e^y}{2} \right) \phi_x \phi_y^2 - 2 \left( \lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)} \right) \phi_x^2 \phi_y + r \phi_x^2 \phi_y^2. \tag{99}$$

The conditions defined in Equation (90) make the Equations (98) and (99) equivalent. Indeed, for this case, we have

$$\lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)} = r - \frac{e^y}{2}. \tag{100}$$

This only suggests that the coefficients involving  $\phi_x \phi_y^2$  and  $\phi_x^2 \phi_y$  are the same under the conditions (90). This occurs because of the symmetric character of the covariant derivative defined in Equation (85). In principle we could have defined the covariant derivative in different ways. If, instead of the definition (85), we use

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \gamma \hat{p}_y, \tag{101}$$

introducing then another parameter  $\gamma$ . In this general case, the symmetric character of the terms linear in  $\hat{p}_x$  and  $\hat{p}_y$  in Equation (98) is not valid anymore, and then, the coefficients involving  $\phi_x \phi_y^2$  and  $\phi_x^2 \phi_y$  would not be necessarily the same. Then, the introduction of  $\gamma$  in general modifies the possible values taken by the parameters of the MG Hamiltonian. No matter which value  $\gamma$  takes in Equation (101); at the most fundamental level, the MG equation emerges from the BS equation after imposing local symmetry under changes in the prices over the BS equation. Still, in the previous calculations, we took  $\gamma = 1$  for the sake of simplicity.

*The Dynamical Origin of the Volatility*

The volatility, entering as a massive term in the Hamiltonian, has a dynamical origin through a mechanism similar to the Higgs one [11,54,55]. In [9], from the general vacuum condition, emerged the relation

$$\phi_{yvac} = \left( \frac{\lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)}}{r - \frac{e^y}{2}} \right) \phi_{xvac}, \tag{102}$$

which, under the condition  $\gamma = 1$  in Equation (101), or equivalently, under the condition (100), gives the result  $\phi_{xvac} = \phi_{yvac}$ . The result (102) is a natural consequence of the general martingale state, defined in Equation (93) [9]. Visualizing the dynamical origin of volatility through the general definition of martingale given in Equation (102) is difficult. Then, for illustrating the generic mechanism, we use a simpler definition of martingale, which is the standard one [4]. In such a case, the martingale state would only depend on the prices of the stock. Considering this, then Equation (99) becomes (ignoring the derivatives with respect to volatility for the martingale state)

$$\langle x | \hat{V}(x, y) | S \rangle = V(S) = -2 \left( r - \frac{e^y}{2} \right) \phi_x \phi_y^2 + r \phi_x^2 \phi_y^2. \tag{103}$$

This result ignores the term  $\langle x | \hat{p}_y | S \rangle = \partial S(x, t) / \partial y = 0$ , since in this special case, we are taking  $S(x, t)$  (martingale state) as a state independent of  $y$ . From Equation (103), we can find that the minimum of the potential gives us a martingale state that only depends on the prices of the stock. This is the ordinary martingale condition that is the same for the BS and MG cases. The ground state in Equation (103) ( $\partial V / \partial \phi_x = 0$ ) gives us the result [9]

$$\phi_{vac} = 1 - \frac{\sigma^2}{2r}. \tag{104}$$

The expansion of the field  $\phi_x$  around the ground state (104) gives

$$\phi(x) = \phi_{vac} + \bar{\phi}(x). \tag{105}$$

This is a shift on the field definition such that the ground state for  $\bar{\phi}(x)$  trivially vanishes. Introducing (105) inside Equation (103) gives us the potential term as a function of the shifted field  $\bar{\phi}(x)$ , as follows:

$$\langle x|\hat{V}(x,y)|S\rangle = V(S) = -2\left(r - \frac{e^y}{2}\right)(\phi_{vac} + \bar{\phi}(x))\phi_y^2 + r(\phi_{vac} + \bar{\phi}(x))^2\phi_y^2. \tag{106}$$

After simplification, the result (106) becomes

$$\langle x|\hat{V}(x,y)|S\rangle = V(S) = \left(-2\left(r - \frac{e^y}{2}\right) + r\phi_{vac}\right)\phi_{vac}\phi_y^2 + \dots \tag{107}$$

In Equation (107), we have only focused on the massive terms for the volatility field. We have not written explicitly the other terms because we want to emphasize the existence of the massive term for the volatility field. From Equation (107), we observe that the dynamical mass for the volatility field is

$$m^2 = \left(-2\left(r - \frac{e^y}{2}\right) + r\phi_{vac}\right)\phi_{vac}. \tag{108}$$

If  $\phi_{vac}$  vanishes (trivial result), then the massive term corresponding to the volatility field vanishes. Then the dynamical origin of the volatility mass emerges from the relation between the parameters  $\sigma$  and  $r$  in Equation (104). The dynamical mechanism for generating the mass for the volatility field, is a non-linear mechanism in this case because of the appearance of  $\sigma$  inside the definition of  $\phi_{vac}$  (Equation (104)) and because of the existence of the term  $e^y$  in the mass definition in Equation (108) (remember that  $e^y = \sigma^2$  for the MG case). Then, the volatility field interacts with itself, generating its own mass. For analyzing the Higgs mechanism in quantum finance, we have used only second-order terms on the polynomial expansion. If we consider higher-order terms in the expansion, still the same arguments used for obtaining the result (107) work. The behavior of the kinetic terms for the MG equation, at all orders, was analyzed in [10].

### 8. Solutions for the Black-Scholes Equation

After analyzing all the details behind the BS equation, it is time to look into its solutions. We follow the solutions as they are illustrated in [2] for this purpose. We denote the prices for the European call option as  $C$  and those for the European put option as  $P$ . The solutions for the BS equation are

$$C = SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{109}$$

for the call option, and

$$P = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \tag{110}$$

for the put option. In Equations (109) and (110),  $N(x)$  corresponds to the standard normal distribution on the  $x$ -variable [56]. In Equations (109) and (110),  $K$  is the strike price,  $T$  is the maturity period for the European option,  $S$  is the price of the stock,  $r$  is the interest rate and  $t$  is the time-variable. The standard normal distributions appear in Equations (109) and (110), depending on the variables  $d_1$  and  $d_2$ , here defined as [2]

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{111}$$

Then,  $d_1$  and  $d_2$  are variables depending explicitly on the parameters  $\sigma$  and  $r$ . It is clear that the volatility cannot be solved explicitly in the BS solution. The best we can do is to

make an iterative process in order to find the value of volatility corresponding to the price of an option, given the values of the interest rate and the variables of the problem [2].

#### *Implied Volatility and Making Decisions Based on the Volatility Estimation*

Implied volatility is just volatility, such that when it is substituted into the Black–Scholes formula, the output is equal to the market price of the option [2,26]. This is the volatility taken from the data available in Yahoo Finance and other websites [25]. The investors then compare the implied value of volatility, with the value of volatility, which they estimate based on the period of time they consider. For example, they could make a daily, monthly or annual estimation of the volatility value. No matter which period of time they select for estimating the volatility, they have to be aware of the fact that the implied volatility and the estimated volatility have to be compared in the same scale of time. From these arguments, it is clear that at the moment of buying an option, the investors normally do not know the value of the volatility. Then, what the investors do is to analyze the historical behavior of the volatility; in other words, they try to see how the volatility changed in time during the previous days, months or years. With such information, the investors then can estimate the volatility, subsequently convert it to some specific scale of time, such that it can be compared with the value taken by the implied volatility. If the estimated volatility is higher than the implied volatility, then the investors should definitely buy the option. Otherwise, if the estimated value is lower than the implied value, then the investors decide not to buy the option. For closing this paper, we illustrate an example taken from [2], and here repeated for clarifying how the investors decide how whether or not they buy an option. Suppose that the value of a European call option on a non-dividend paying stock is 3.67 USD when  $S = 33$ ,  $K = 30$ ,  $r = 0.05$  and  $T - t = 0.25$  years (three months). The implied volatility would be the value of  $\sigma$ , such that when we introduce it inside the BS solution in Equation (109), together with the given values for  $S$ ,  $K$ ,  $r$  and  $T - t$ , it gives us as a result the value for the call option  $C = 3.67$  USD per share. It is not difficult to realize that the process is iterative. The result for this example gives a volatility of 22% per annum. This result is taken from data in the charts, after solving  $\sigma$  in Equation (109). On the other hand, let us assume that the investor thinks that the volatility will be 1.5% per day. Then we have to convert this daily estimation to an annual value. The formula to apply is

$$\sigma_{annual} = \sigma_{day} \sqrt{D_{year}}, \quad (112)$$

where  $D_{year}$  is the number of trading days in a year. If we use this formula, then we obtain

$$\sigma_{annual} = 1.5\sqrt{256} = 24\%, \quad (113)$$

where we have assumed that there are 256 trading days in a year (Taking into account that the trading cannot occur everyday). From the standard theory we know that the higher the volatility of an option is, the higher is its value. Since the investor estimates a volatility of  $\sigma_{estimated} = 24\%$ , which is a larger value than the estimated volatility, then he will certainly invest in the option value.

## 9. Conclusions

In this paper, we have revised some fundamental properties of the BS equation. We have also expressed its Hamiltonian formulation, which facilitates the analysis of the dynamics of the option market. We then explained some fundamental mechanisms, suggesting the spontaneous symmetry breaking and Higgs mechanism inside the scenario of quantum finance, by considering symmetries under changes of the prices of stocks. If we impose the local symmetry with respect to the changes in the prices over the BS equation, then the MG equation emerges naturally as a gauge-theory. This seminal result is a natural consequence of extending the concept of gauge theories to the scenario of quantum finance. In this paper, we also analyzed the flow of information in the stock market, and finally, we explained how the investors use the solutions of the BS equation for



making investment decisions by comparing the estimated volatility with the implied value of volatility. Implied volatility is just the value of volatility obtained from the BS equation after replacing all the other variables and parameters observed from the stock charts. On the other hand, estimated volatility is the value of volatility, which the investor believes should be the right one. Since the value of an option increases with the volatility; when an investor estimates a value of volatility larger than the implied value of volatility, then he/she should invest in the option. However, if the estimated value is smaller than the implied value of volatility, then the investor should decide not to invest at all in the option. In general, the option market is a zero-sum game between the holder of an option and the writer of the same option. This interesting game is modeled through the BS equation, which helps the investors to make smart decisions. We must remark that although the Black-Scholes has certain limitations, such as, for example, the fact that the volatility appears as a parameter instead of appearing as a field or variable; still, several investors use it for making investment decisions. The key power of the equation is its capability of canceling random fluctuations on the prices of the Options and its simplicity. Still, it can be complemented with the Merton–Garman equation in order to develop models for analyzing the volatility. In this paper we simply limited ourselves to mention the Merton–Garman equation, and we then explained its local equivalence with the Black–Scholes equation. This part is important because it touches on the most difficult parameter to predict from the BS equation, namely, the volatility. Finally, we must remark that the BS equation predicts the fair price of options for both call options and put options. For the call options, the holders want the prices of the same options to increase in time, because the price they agreed on for buying the option is predetermined. This is the case because the holder of a call option has the right to buy the option at the initially agreed price. For the put option, the situation is different, namely, the holder has the right to sell it at an initially determined price. Then the holder of the put option wishes the prices of the option would decrease in time, such that he/she can sell the option at a higher price than the price at maturity. It is for this reason that the Holders of a call option are called bullish and the holders of the put option are called bearish.

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