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The Hawking Radiation in Massive Gravity: Path Integral and the Bogoliubov Method

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Abstract: We prove the consistency of the different approaches for deriving the black hole radiation for the spherically symmetric case inside the theory of Massive Gravity. By comparing the results obtained by using the Bogoliubov transformations with those obtained by using the Path Integral formulation, we find that in both cases, the presence of the extra-degrees of freedom creates the effect of extra-particles creation due to the distortions on the definitions of time defined by the different observers at large scales. This, however, does not mean extra-particle creation at the horizon level. Instead, the apparent additional particles perceived at large scales emerge from how distant observers define their time coordinate, which is distorted due to the existence of extra-degrees of freedom.

Keywords: Massive Gravity; Hawking radiation; Path Integral method; Bogoliubov transformation method; extra-degrees of freedom



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1. Introduction

The theory of general relativity (GR) predicts the existence of black holes. The classical theory suggests that no object can escape from a black hole once it crosses the event horizon. In this way, although the thermodynamics of black holes was developed a long time ago, it was believed by then that these objects could not emit radiation [1–4]. However, Hawking demonstrated in a seminal paper that quantum effects can make the black holes evaporate by emitting particles at a rate defined by the surface gravity [5]. In its original derivation, Hawking used the method of Bogoliubov transformations to compare two different vacuums, one located at the future null infinity and the other one located at the past null infinity. The effect of particle creation then appeared as a consequence of the mix of positive and negative frequencies. The particle emission process was also proven using the Path Integral method, where the periodicity of the poles of the propagators of a scalar field is equivalent to the effect of particle creation [6]. In this paper, we make the two derivations of the black hole temperature as it is perceived by observers in Massive Gravity. In both methods, there appear modifications of the surface gravity due to the presence of the extra-degrees of freedom entering as a distortion of the notion of time in the theory. Note that the modifications appear as a consequence of the way how the observers define the notion of time with respect to the preferred time direction defined by the Stückelberg function $T_0(r, t)$, which contains the effect of the extra-degrees of freedom of the theory. This means that the fact that observers located at large scales in Massive Gravity define a different surface gravity with respect to observers in GR does not mean that there is an extra-particle creation at the horizon level. Instead, what is happening, in reality, is that since the notion of the particle depends on the way how we define the positive frequencies, and simultaneously, the definition of a positive frequency depends on the way how we

define the time coordinate, then the disagreement in the number of particles between GR and Massive Gravity comes out from the fact that the extra-degrees of freedom affect and distort the time coordinate in Massive Gravity. In other words, an observer in GR located at large scales from a black hole will define the time in a different way in comparison with an observer located at the same scale but operating inside the theory of Massive Gravity. The definition of time naturally affects the definition of frequency. The paper is organized as follows: In Section 2, we briefly review the most generic black hole solution inside the scenario of Massive Gravity. In Section 3, we revise the Bogoliubov transformation method applied to Massive Gravity to calculate the number of particles perceived by an observer located at a large distance with respect to the black hole. In Section 4, we develop the Path Integral method for analyzing the same amount of particles perceived by an observer located at large scales with respect to the center of the black hole. We then proceed to compare the results with those obtained via Bogoliubov transformations. Finally, in Section 5, we conclude.

2. The Black Hole Solution in Massive Gravity

The black hole solutions in Massive Gravity can be obtained after solving the field equations which come from the massive action

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g}(R + m_g^2 U(g, \phi)) \tag{1}$$

Here, $U(g, \phi)$ is the potential term, and it is defined as

$$U(g, \phi) = U_2 + \alpha_3 U_3 + \alpha_4 U_4. \tag{2}$$

Here, α_3 and α_4 are the two free parameters of the theory. The definitions for each order of the potential $U_n(g, \phi)$ can be found in [7–9]. The field equations are then

$$G_{\mu\nu} = -m_g^2 X_{\mu\nu}, \tag{3}$$

with the energy-momentum tensor given by

$$X_{\mu\nu} = \frac{\delta U}{\delta g^{\mu\nu}} - \frac{1}{2} U g_{\mu\nu}. \tag{4}$$

In Equation (3), m_g corresponds to the graviton mass. The spherically symmetric solutions for the previous field equations are obtained as

$$ds^2 = G_{tt} dt^2 + G_{rr} S_0^2 dr^2 + G_{rt}(drdt + dt dr) + S_0^2 r^2 d\Omega_2^2, \tag{5}$$

where

$$\begin{aligned} G_{tt} &= -f(S_0 r)(\partial_t T_0(r, t))^2, & G_{rr} &= -f(S_0 r)(\partial_r T_0(r, t))^2 + \frac{1}{f(S_0 r)}, \\ G_{tr} &= -f(S_0 r)\partial_t T_0(r, t)\partial_r T_0(r, t). \end{aligned} \tag{6}$$

Here, $f(S_0 r) = 1 - \frac{2GM}{S_0 r} - \frac{1}{3}\Lambda(S_0 r)^2$, with M being the mass of the black hole, Λ being a constant and S_0 being a parameter to be defined here. In a compact form, the spherically symmetric solutions can be found to be

$$ds^2 = -f(S_0 r)dT_0(r, t)^2 + \frac{S_0^2 dr^2}{f(S_0 r)} + S_0^2 r^2 d\Omega^2, \tag{7}$$

working in unitary gauge [7]. In this metric, the Stückelberg function operates as a preferred direction of time, different in general from the ordinary time coordinate direction t . Then it is necessary to define two different time-like Killing vectors, one in the direction $T_0(r, t)$ and another one pointing in the direction of the ordinary time coordinate t . This mismatch between the directions of the two Killing vectors will generate a difference between the

number of particles perceived by observers defined in Massive Gravity and the number of particles perceived by observers satisfying the same conditions of motion in GR [10,11]. In general, it is known that

$$T_0(r, t) = S_0 t + A(r, t). \tag{8}$$

Here, $T_0(r, t)$ behaves as a preferred time-direction [12–14]. S_0 is a scale factor depending on the two free parameters of the theory [7]. However, in this paper, we still focus on the case where the relation $\beta = \alpha^2$ is satisfied. This reduces the number of free parameters to only one, and then S_0 is defined as

$$S_0 = \frac{\alpha}{1 + \alpha}. \tag{9}$$

Here, the connection between α and β with the two free-parameters α_3 and α_4 given in Equation (2) is the following [7]

$$\alpha = 1 + 3\alpha_3, \quad \beta = 3(\alpha_3 + 4\alpha_4). \tag{10}$$

Then in this paper, we will analyze the particle creation process of the black hole solution in Massive Gravity for the solution (7) and for the case where we only have one free parameter satisfying the condition $\beta = \alpha^2$.

3. The Bogoliubov Transformation Method in Massive Gravity: Hawking Radiation

In the Bogoliubov transformation method, we have to define a couple of vacuums. Both vacuums will define a different Stückelberg function and, consequently, a different value for the function $A(r, t)$. Then both vacuums will be inequivalent in general. This inequivalence between the pair of vacuums under study can be perceived by the observers in Massive Gravity as a particle creation effect. However, as was the case when we analyzed the Path Integral formulation, the amount of particles emitted by the black hole at the event horizon does not change in this case with respect to the situation analyzed in GR. However, the fact that the extra-degrees of freedom create the distortion effect can make the observers located at large scales believe that there are extra particles emitted by the black hole. In fact, this is just an illusion in the sense that there are no extra-particles coming from the horizon. However, the effect is real in the sense that the distortion of time is equivalent to a distortion of the notion of vacuum, and then the observer’s detectors will really perceive an extra-component of radiation. The results obtained for the observers defining the time coordinate in agreement with $T_0(r, t)$ will not differ with respect to the results reported by observers in GR. Then we can use the standard and well-known Penrose diagrams if we use the transformed Stückelberg functions $U(r, T_0(r, t)) = u + T_0(r, t)$ and $V(r, T_0(r, t)) = v + T_0(r, t)$. Then the causal structure of the spacetime defined with respect to $T_0(r, t)$ will be the same as in GR. Without loss of generality, we will take the spacetime.

Defined with respect to $T_0(r, t)$ as asymptotically flat. The asymptotically flat diagram can be seen in Figure 1. The deviations with respect to the usual notion of time due to the presence of the extra-degrees of freedom have to be appreciable enough to create distortions of time over the observers located at large scales and define the time arbitrarily. This distortion will affect the way how the particles are defined in the theory of Massive Gravity and possibly the causal structure of spacetime. If we define on J^+ the vector n^a , which is a future-directed null vector at x , pointing radially inward. Then the vector $-\epsilon n^a$ joins the future event horizon with a surface of constant $U(r, T_0(r, t))$. We then define another null vector tangent to the horizon such that the normalization $n^a l_a = -1$ is valid. Here, we will demonstrate that the presence of the extra-degrees of freedom in massive gravity creates a distortion that will modify the relation between the coordinate u and the affine parameter $-\epsilon$ [5]. After doing the corresponding parallel transport of the vectors, we obtain the relation

$$\epsilon = C e^{\kappa U(r, T_0)}, \tag{11}$$

between the affine parameter and the Stückelberg function $U(r, T_0(r, t))$. To understand the previous relation, we have to define the scalar field moving around the black hole by expanding it in terms of positive and negative frequencies as

$$\phi = \sum_i (f_i \hat{a}_i + \bar{f}_i \hat{a}_i^+). \tag{12}$$

Here, the solutions for f_i form a complete set of orthonormal functions over the past infinity J^- . Then they contain positive frequencies only with respect to the canonical affine parameter on J^- [5]. In the coming analysis, we will need to define the orthonormal condition, given by

$$\frac{1}{2} \int_S (f_i \bar{f}_{j;a} - \bar{f}_j f_{i;a}) d\Sigma^a = \delta_{ij} \tag{13}$$

Here, the integration is performed over a suitable surface S . The upper bar over the functions means complex conjugation operation. We remark that the functions f_i have all the Cauchy data defined over the past infinity. Then the operators \hat{a}_i and \hat{a}_i^+ have the natural interpretation of annihilation and creation operators for particles over the past infinity (incoming particles). We can define everywhere the field in the form given by its expansion with respect to the functions f_i . However, it is also possible to expand the same field as follows

$$\phi = \sum_i (p_i \hat{b}_i + \bar{p}_i \hat{b}_i^+ + q_i \hat{c}_i + \bar{q}_i \hat{c}_i^+). \tag{14}$$

Here, the functions p_i have zero Cauchy data at the future event horizon, and they represent outgoing components. They form an orthonormal family over the surface J^+ , and they only have positive frequencies with respect to the affine parameter along the null geodesic generators on J^+ . Then the operators \hat{b}_i and \hat{b}_i^+ represent the annihilation and creation operators for particles on the future infinity J^+ . On the other hand, the functions q_i have zero Cauchy data at the future infinity J^+ . They form a complete set of orthonormal functions over the event horizon. However, it is not possible to define, in this case, a region over which we can define positive frequencies and then the meaning of the operators \hat{c}_i and \hat{c}_i^+ is not clear for this case, although not important at the moment of doing the calculations of the black hole radiation. What is important at this point is that the scalar field can either be expanded in terms of the functions f_i or in terms of the functions p_i and q_i . To keep the canonical commutation relations, the previous functions, as well as the operators, must be related to each other through the Bogoliubov transformations defined as follows [5].

$$\begin{aligned} p_i &= \sum_j (\alpha_{ij} f_j + \beta_{ij} \bar{f}_j), \\ q_i &= \sum_j (\gamma_{ij} f_j + \eta_{ij} \bar{f}_j), \end{aligned} \tag{15}$$

for the functions and

$$\begin{aligned} \hat{b}_i &= \sum_j (\bar{\alpha}_{ij} \hat{a}_j + \bar{\beta}_{ij} \hat{a}_j^+), \\ \hat{c}_i &= \sum_j (\bar{\gamma}_{ij} \hat{a}_j + \bar{\eta}_{ij} \hat{a}_j^+), \end{aligned} \tag{16}$$

for the annihilation operators. The creation operators can be obtained by applying the adjoint operation over the previous operators. It is then clear that the fact that we have an initial vacuum with no particles $\hat{a}_i|0\rangle = 0$, does not guarantee that other vacuums defined in other locations of spacetime share the same definition of particle. This happens when $\beta_{ij} \neq 0$, which is the coefficient mixing the positive and negative frequencies. Then

the observers located at the future infinity J^+ will perceive particle creation. In fact, the expectation value of the number operator defined by using the operators \hat{b}_i is

$$\langle 0 | \hat{b}_i^+ \hat{b}_j | 0 \rangle = \sum_j |\beta_{ij}|^2 \tag{17}$$

This just corresponds to the standard understanding of the concept of a particle in asymptotically flat spacetime [15]. In what follows, we will divide the analysis into two parts, namely, the case of GR, which is equivalent to saying that the observers in Massive Gravity take the time coordinate as $T_0(r, t)$, and the other case is when the observers assume an arbitrary direction for the time coordinate. The way how observers define their local time depends on their conditions of motion.

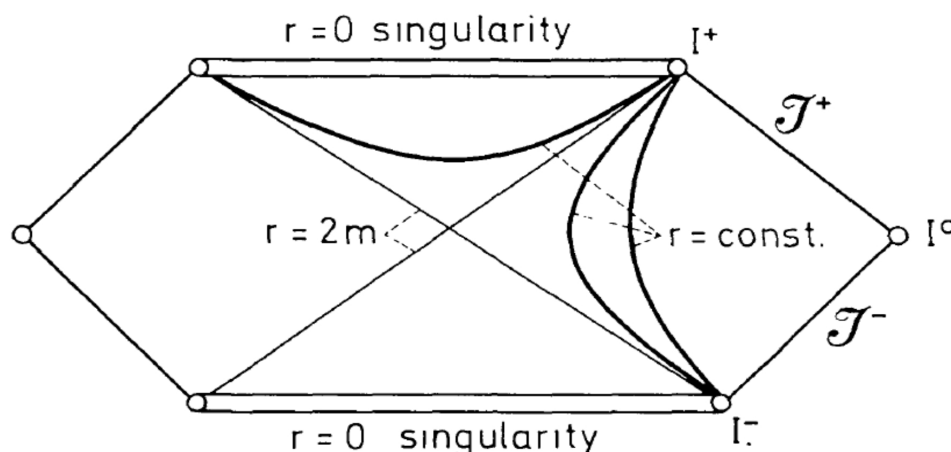


Figure 1. The Penrose diagram for the Schwarzschild geometry in GR as is shown in [5]. In Massive Gravity, the same diagram is valid if we express the black hole solution in terms of the Stückelberg functions. The past-null infinity (J^-) of the diagram corresponds to the event where the black hole has not yet formed. The future null infinity (J^+), on the other hand, corresponds to the case where the black hole is already formed.

3.1. The Case of GR: Observers Defining the Time in Agreement with $T_0(r, t)$ in Massive Gravity

For this case, we can use safely the relation (11). In such a case, it is easy to see that the surfaces of constant phase $\omega U(r, T_0(r, t))$ are defined as

$$\omega U = -\frac{\omega}{\kappa} (\log \epsilon - \log C). \tag{18}$$

On the past infinity J^- , the Killing vector K^a is parallel to the vector n^a . Then we can assume a relation $n^a = DK^a$, which just rescales the time coordinate and then the phase of the solution in the past infinity is given by

$$-\frac{\omega}{\kappa} (\log(V_0 - V) - \log D - \log C). \tag{19}$$

Here again, V corresponds to the light-like Stückelberg function, and it depends explicitly on r and $T_0(r, t)$. The solutions corresponding to the Fourier components, expressed in spherically symmetric form and in terms of advanced and retarded time (Stückelberg functions) are defined as

$$\begin{aligned} f_{\omega', l, m} &= (2\pi)^{-1/2} r^{-1} (\omega')^{-1/2} F_{\omega'}(r) e^{i\omega' V} Y_{l, m}(\theta, \phi), \\ p_{\omega, l, m} &= (2\pi)^{-1/2} r^{-1} (\omega)^{-1/2} P_{\omega}(r) e^{i\omega U} Y_{l, m}(\theta, \phi). \end{aligned} \tag{20}$$

Here, $Y_{l, m}(\theta, \phi)$ are the spherical harmonics normalized in the standard form. Due to the Bogolibov transformations, we can express the functions p_i as a linear combination of the

functions f_i , as has been explained previously. This is possible after doing the integration over the frequency ω' as follows

$$p_\omega = \int_0^\infty d\omega' (\alpha_{\omega,\omega'} f_{\omega'} + \beta_{\omega,\omega'} \bar{f}_{\omega'}). \tag{21}$$

Here, we ignore the indices corresponding to the spherical harmonics, understanding that formally they must appear. However, their presence will not contribute to the physics developed in this section. If we replace the phase (19) inside the solution for the function $p_{\omega,l,m}$ defined in Equation (20), then we get

$$p_\omega^{(2)} \approx (2\pi)^{-1/2} r^{-1} (\omega)^{-1/2} P_\omega^- \left(\frac{V_0 - V}{CD} \right)^{-i \frac{\omega}{\kappa_{eff}}}. \tag{22}$$

If we make a Fourier transformation with respect to V , it is trivial to demonstrate that the Bogoliubov coefficients $\alpha_{\omega,\omega'}^{(2)}$ and $\beta_{\omega,\omega'}^{(2)}$ are defined for large values of ω' as

$$\begin{aligned} \beta_{\omega,\omega'}^{(2)} &\approx -i \alpha_{\omega,(-\omega')}^{(2)}, \\ \alpha_{\omega,\omega'}^{(2)} &\approx (2\pi)^{-1} P_\omega^- (CD)^{i \frac{\omega}{\kappa_{eff}}} e^{i(\omega-\omega')v_0} \left(\frac{\omega'}{\omega} \right)^{1/2} \times \\ &\quad \Gamma \left(1 - \frac{i\omega}{\kappa} \right) (-i\omega')^{-1+i \frac{\omega}{\kappa_{eff}}} \end{aligned} \tag{23}$$

Here, $\alpha^{(2)}$ can be obtained from $\beta^{(2)}$ if we make an analytical continuation of $\beta^{(2)}$ around the logarithmic singularity is defined by the term $(-i\omega')^{-1+i \frac{\omega}{\kappa_{eff}}}$ in the previous results. In fact, if we round the logarithmic singularity in the neighborhood of $\omega' \rightarrow 0$ by taking $\omega' \rightarrow e^{i\pi} \omega'$, then we obtain

$$|\alpha_{\omega,\omega'}^{(2)}| = e^{\frac{\pi\omega}{\kappa_{eff}}} |\beta_{\omega,\omega'}^{(2)}|. \tag{24}$$

This result corresponds to the standard one derived by Hawking [5], and it will define the number of particles that an observer defining the time in agreement with $T_0(r, t)$ will perceive. In this part of the paper, we have defined the surface gravity as κ_{eff} . This quantity will be defined as the effective surface gravity perceived by the observers defining the time in agreement with $T_0(r, t)$. Observers defining the time in a different way will perceive a different κ which can be connected functionally with κ_{eff} .

3.2. The Case of Observers Defining the Time Arbitrary

In this subsection, we will derive the surface gravity κ for observers defining the time arbitrarily by calculating first the effective surface gravity κ_{eff} as a function of κ . Once again, we remark that κ_{eff} corresponds to the standard result perceived by the observers defining the time $T_0(r, t)$. We can repeat the previous reasoning for the case of observers defining the time arbitrarily, then some changes will appear due to the presence of the function $A(r, t)$ inside the advanced and retarded light-like Stückelberg functions $U(r, T_0(r, t))$ and $V(r, T_0(r, t))$. Here, $T_0(r, t)$ is the standard Stückelberg function. By taking into account that $T_0(r, t) = S_0t + A(r, t)$ and rescaling the time coordinate as $t \rightarrow S_0t$ [7]; then the surfaces of the constant phase defined previously become

$$\omega u = -\frac{\omega}{\kappa} (\log \epsilon - \log C) - \omega A(r, u). \tag{25}$$

This redefinition of the surfaces of the constant phase depends on $A(r, u)$, which is the distortion of time created by the extra-degrees of freedom at large scales. This function, $A(r, u)$, depends explicitly on the variable u . This point will be important at the moment of calculating the black hole radiation perceived by the observers defining the time arbitrarily.

Note that it is expected the function $A(r, u)$ to vanish in the neighborhood of the future event horizon if GR is recovered for strong gravitational fields, as the standard theory of Massive Gravity suggests. We can express the contribution of $A(r, u)$ in a different way, such that the result looks like

$$\omega u = -\frac{\omega}{\kappa}(\log \epsilon - \log C - \log(e^{\kappa A(r,u)})). \tag{26}$$

Here again on the past infinity, J^- , the Killing vector K^a is parallel to the vector n^a . Then we can assume the same relation $n^a = DK^a$ and then we get

$$-\frac{\omega}{\kappa}(\log(V_0 - V)) - \log D - \log C - \log(e^{\kappa A(r,u)}), \tag{27}$$

in close analogy to what happens in [5]. In Equation (27), V is defined as $V = v + A(r, v)$. However, near the past event horizon, it is also expected the extra-degrees of freedom to become negligible and then $V \approx v$. This approximation is not valid at scales far away from the past event horizon, and then we will keep in mind the fact that we have to include the function $A(r, v)$ in the calculations. A detail to remark on is that here we are taking $A(r, u)$ as the function related to $U(r, T_0(r, t)) = u + A(r, u)$. This is the case because here, we define the retarded time as given by u . On the other hand, we also define $V(r, T_0(r, t)) = v + A(r, v)$ by defining, in this case, the advanced time as v . The relation between u and v is defined by the result (19). Along the past infinity, the solution will be

$$p_\omega^{(2)} \sim (2\pi)^{-1/2} \omega^{-1/2} r^{-1} P_\omega^- \exp\left(-\frac{i\omega}{\kappa} \log\left(\frac{v_0 - v - A(r, v)}{CD e^{\kappa A(r, u)}}\right)\right) \tag{28}$$

The difference between the standard case and the Massive Gravity one is the presence of the term $A(r, u)$, which will affect the integration over the variable v at the moment of doing the Fourier transformation to find the Bogoliubov coefficients. Later, we will see that the term $A(r, v)$ does not affect the integration over v after doing the appropriate substitution of variables. The case of $A(r, u)$ is different due to the non-trivial relation between u and v . However, the fact that we still have a black hole emitting particles in agreement with the Fermi–Dirac statistic remains. The fraction of particles entering the black hole is given by [5]

$$\Gamma_{jn} = \int_0^\infty (|\alpha_{\omega, \omega'}^{(2)}|^2 - |\beta_{\omega, \omega'}^{(2)}|^2). \tag{29}$$

To find the Bogoliubov coefficient $\alpha^{(2)}$, then we have to multiply the result (29) by \bar{f}_{ja} here defined by

$$\bar{f}_{\omega', v} = -i(1 + \partial_v A(r, v))(2\pi)^{-1/2} r^{-1} (\omega')^{1/2} F_{\omega'}(r) e^{-i\omega'(v+A(r, v))}. \tag{30}$$

This result is obtained from Equation (20) if we take the complex conjugate for the function $f_{\omega'}$ and we take the derivative with respect to v . Here, we have used the definition $V = v + A(r, v)$ and we have also ignored the spherical harmonics contribution, which we assume to be normalized in the standard way. By multiplying the result (28) with the previous result and then integrating over the variable v to make the Fourier transformation, we obtain

$$\alpha_{\omega, \omega'}^{(2)} \approx -(2\pi)^{-1} P_\omega^- (CD)^{i\frac{\omega}{\kappa}} \left(\frac{\omega'}{\omega}\right)^{-1/2} \times \int (v_0 - v - A(r, v))^{-i\frac{\omega}{\kappa}} e^{i\omega A(r, u)} (1 + \partial_v A(r, v)) e^{i\omega'(v+A(r, v))} dv. \tag{31}$$

Here, $V_0 \approx v_0$ since v_0 has a correspondence with the past event horizon by assuming that the whole Penrose diagram corresponds to the vacuum Schwarzschild solution. Here, the integration is performed for all the values of v . In this case, in general, we do not get the same Gamma function obtained previously inside the standard calculations. Instead, the

result in Equation (31) depends on the functional behavior of $A(r, u)$. It is trivial to observe that the result will not depend on $A(r, v)$. If we make the replacement $z = v_0 - v - A(r, v)$, in the previous integral, then we get

$$\alpha_{\omega, \omega'}^{(2)} \approx -(2\pi)^{-1} P_{\omega}^{-} (CD)^{i\frac{\omega}{\kappa}} \left(\frac{\omega'}{\omega}\right)^{-1/2} \times \int (z)^{-i\frac{\omega}{\kappa}} e^{i\omega A(r, u(z))} e^{-i\omega'(v_0-z)} dz. \tag{32}$$

Here, we make one additional replacement by taking $x = -i\omega'z$, getting then the result

$$\alpha_{\omega, \omega'}^{(2)} \approx -(2\pi)^{-1} P_{\omega}^{-} (CD)^{i\frac{\omega}{\kappa}} e^{-i\omega'v_0} \left(\frac{\omega'}{\omega}\right)^{-1/2} \times (-i\omega')^{-1+i\frac{\omega}{\kappa}} \int (x)^{-i\frac{\omega}{\kappa}} e^{i\omega A(r, u(x))} e^{-x} dx. \tag{33}$$

For the case when $A(r, u) = 0$, we recover the result (23). Here, however, we will take the function $A(r, u)$ as a polynomial expansion on the variable u as follows

$$A(r, u) \approx \sum_{n=0}^{\infty} a_n u^n. \tag{34}$$

In addition, we have to take into account the relation between u and v already obtained in Equation (19). Then here we use

$$A(r, u) \approx \sum_{n=0}^{\infty} a_n \left(-\frac{1}{\kappa} \log\left(\frac{v_0 - v - A(r, v)}{DC}\right)\right)^n \tag{35}$$

Then the contribution of $A(r, u)$ to the integral in Equation (33) is

$$e^{i\omega A(r, u(x))} = \left(\frac{v_0 - v - A(r, v)}{CD}\right)^{i\omega \sum_n \frac{na_n}{\kappa^n}}. \tag{36}$$

If we make the same changes of variables as before, this previous expression is given by

$$e^{i\omega A(r, u(x))} = (-i\omega')^{i\omega \sum_n \frac{na_n}{\kappa^n}} (DC)^{i\omega \sum_n \frac{na_n}{\kappa^n}} (x)^{-i\omega \sum_n \frac{na_n}{\kappa^n}} \tag{37}$$

In the previous series, the coefficient a_1 , corresponding to $n = 1$, can be defined in a convenient way, such that it is possible to write the result (33) as

$$\alpha_{\omega, \omega'}^{(2)} \approx -(2\pi)^{-1} P_{\omega}^{-} (CD)^{i\omega \sum_n \frac{na_n}{\kappa^n}} e^{-i\omega'v_0} \left(\frac{\omega'}{\omega}\right)^{-1/2} \times (-i\omega')^{-1+i\omega \sum_n \frac{na_n}{\kappa^n}} \int (x)^{-i\omega \sum_n \frac{na_n}{\kappa^n}} e^{-x} dx. \tag{38}$$

The result is simply given by

$$\beta_{\omega, \omega'}^{(2)} \approx -i\alpha_{\omega, (-\omega')}^{(2)},$$

$$\alpha_{\omega, \omega'}^{(2)} \approx (2\pi)^{-1} P_{\omega}^{-} (CD)^{i\omega \sum_n \frac{na_n}{\kappa^n}} e^{i(\omega-\omega')v_0} \left(\frac{\omega'}{\omega}\right)^{1/2} \times \Gamma\left(1 - i\omega \sum_n \frac{na_n}{\kappa^n}\right) (-i\omega')^{-1+i\omega \sum_n \frac{na_n}{\kappa^n}} \tag{39}$$

Here, the relation between $\alpha^{(2)}$ and $\beta^{(2)}$ is not modified. In fact, if we make the analytical continuation around the logarithmic singularity defined this time by the term

$(-i\omega')^{-1+i\omega \sum_n \frac{na_n}{\kappa^n}}$, to get $\alpha^{(2)}$ and $\beta^{(2)}$ by using $\omega' \rightarrow e^{i\pi}\omega'$, then we obtain the relation

$$|\alpha_{\omega,\omega'}^{(2)}| = e^{\pi\omega \sum_n \frac{na_n}{\kappa^n}} |\beta_{\omega,\omega'}^{(2)}|. \tag{40}$$

This has a correspondence to the result (24). Then the total number of particles created in a given mode is defined by the relation

$$|\beta_{\omega,\omega'}^{(2)}| \approx \Gamma_{\omega'} \left(e^{2\pi\omega \sum_n \frac{na_n}{\kappa^n}} - 1 \right)^{-1}, \tag{41}$$

still consistent with the statistics followed by the black body radiation. From this previous result, however, we can define an effective surface gravity given by

$$\kappa_{eff} = \left(\sum_n \frac{na_n}{\kappa^n} \right)^{-1} = \frac{1}{4GM}. \tag{42}$$

Here, κ is the surface gravity perceived by an observer defining the time in an arbitrary way. On the other hand, κ_{eff} is the surface gravity perceived by an observer defining the time in agreement with $T_0(r, t)$. Here, we have used some specific functional dependence for $A(r, u)$, and the result can change depending on how this function behaves. The method developed is, however, general, and the functional behavior selected is a polynomial expansion which is what should be expected. The radial dependence of this function is irrelevant since this variable will not enter the integration over v . If we want to find the surface gravity perceived by the observers defining the time arbitrarily, we have to solve Equation (42) for κ . It is evident that the result will be different from the standard one. Before moving forward with the Path Integral method, we must remark that the Bogoliubov method studies the gravitational collapse of the black hole. This is the case because the past null infinity J^- represents the vacuum state when there is no black hole or the standard vacuum state without particles. However, the future null-infinity J^+ represents the case where the black hole is already formed, and although locally, an observer can still define a vacuum state, when we compare the vacuum state at J^- , with the vacuum state at J^+ , then we find that there are a certain amount of particles emerging at J^+ for the corresponding observers. It is for this reason that the Penrose diagrams are representations of the gravitational collapse.

4. The Path Integral Formulation of the Black Hole Radiation in Massive Gravity

The Path Integral formulation for evaluating the black hole temperature in Massive Gravity was developed in [10,11]. The result suggested that the periodicity of the poles of the propagator for the scalar field is affected by the presence of the extra-degrees of freedom. This is consistent with the fact that $T_0(r, t)$ behaves as a preferred direction of time, and in general, the analytical extension of $T_0(r, t)$ will differ from the analytical extension defined for the ordinary time coordinate t . The condition of regularity for the Cauchy data means that the following result must be satisfied [10,11]

$$-4\pi GM < \psi(r, t) < 0. \tag{43}$$

Here, $\psi(r, t) = \mu + \bar{A}(r, t)$, where we have defined the time coordinate as $t = \gamma + i\mu$, separating it in the real and imaginary parts. In addition, $A(r, t) = Re(A(r, t)) + i\bar{A}(r, t)$, having this function real and imaginary part as well. We can notice that the imaginary part of the time coordinate defines the periodicity of the propagator. The periodicity will be affected for the observers defining the time arbitrarily, if the function $A(r, t)$ has an imaginary component, namely, if $\bar{A}(r, t) \neq 0$. The result (43) can then be expressed more explicitly as

$$-4\pi GM < \mu + \bar{A}(r, t) < 0. \tag{44}$$

Then the number of particles perceived by an observer in Massive Gravity will depend on how he/she defines the local time. If an observer defines the time in agreement with $T_0(r, t)$, then the number of particles perceived will be the same as in the GR case. The temperature perceived by the observers defining an arbitrary direction of time t will depend on the explicit solution for μ coming from the condition of periodicity of the poles of the propagator

$$8\pi GM = \mu + \bar{A}(r, t). \tag{45}$$

Then the problem is reduced in finding the solution for μ from this previous expression. Here, $\bar{A}(r, t)$ can have any dependence. However, without loss of generality, we can take the function $\bar{A}(r, t)$ to be a polynomial expansion of μ , $\bar{A}(r, t) = \sum_{n=0}^{\infty} b_n \mu^n$. Here again, the linear term in the expansion can absorb the linear term μ , and then we can express the result as

$$8\pi GM = \sum_{n=0}^{\infty} b_n \mu^n = \mu_{eff}. \tag{46}$$

Here, μ_{eff} is the complex component of time defined by the observers defining the time in agreement with $T_0(r, t)$. On the other hand, μ is related to the observers defining the time arbitrarily. Then we can define the black hole temperature as

$$T_{eff} = \frac{1}{\mu_{eff}} = \frac{1}{8\pi GM} = \frac{1}{\sum_n b_n \mu^n}. \tag{47}$$

In this way, we can see that the observers defining the time arbitrarily will, in general, perceive a different temperature with respect to the standard one calculated in GR. Order by order, there is a direct correspondence between this previous result and the one obtained in Equation (42). This can be seen after taking into account the well-known relation between surface gravity and temperature here expressed as

$$T = \frac{1}{2\pi} \kappa = \frac{1}{\mu}. \tag{48}$$

This result is the temperature perceived by observers defining the time arbitrarily. μ can be found by solving the polynomial Equation (46). The solution is, in general, non-trivial. Equation (47) has a direct relation with the expansion performed in Equation (42), where the Bogoliubov methods were used.

Further Analysis

To see that the method used here is general, we will analyze the functional dependence of $\bar{A}(r, \gamma + i\mu)$. In [10,11], it was assumed that it was always possible to find $\bar{A}(r, t)$, and then the surface gravity was defined for the simplest case. Here, however, we go deeper into the analysis to explain that our previous result is general. For simplicity, to illustrate the consistency, we will assume that $\mu \ll t$, namely, that the imaginary component of the analytically extended time coordinate is much smaller than the real component. In such a case we can define the expansion

$$A(r, \gamma + i\mu) \approx \sum_{n=even} \left(n! \partial_{i\mu}^{(n)} A(r, t)_{\mu=0} \right) \cos \mu + i \sum_{n=odd} \left(n! \partial_{i\mu}^{(n)} A(r, t)_{\mu=0} \right) \sin \mu \tag{49}$$

Then in this case we define the analytically extended Stückelberg function as

$$\bar{A}(r, t) \approx \sum_{n=odd} \left(n! \partial_{i\mu}^{(n)} A(r, t)_{\mu=0} \right) \sin \mu. \tag{50}$$

From this definition, it is evident that the analytical extension of $T_0(r, t)$ will be different from the analytical extension of t . This is the mismatch that makes the concept of particle ambiguous in Massive Gravity for observers located at large scales. This distortion effect is absent in GR, and then it is a consequence of the extra-degrees of freedom. Whenever the time is distorted, the notion of the particle is modified with respect to the same observers defined in GR. This will generate the effect of particle creation for observers located at large scales in Massive Gravity. However, this does not mean that there are extra-particles coming from the event horizon because the effects come from the ambiguity of the vacuum definitions generated by the extra-degrees of freedom of the theory. This is an interesting effect, and it helps us to understand that not only the curvature effects make it possible to perceive Hawking radiation, but also any other contribution able to create distortions in the concept of time will generate Hawking radiation. This effect is general, and it might appear whenever there are degrees of freedom or any other physical effect able to affect the way how we define time at different scales. Going further into the previous calculations, we can then express the analyticity condition for the propagator as

$$-4\pi GM < \mu + \sum_{n=odd} \left(n! \partial_{i\mu}^{(n)} A(r, t)_{\mu=0} \right) \sin \mu < 0 \tag{51}$$

The associated periodicity condition will be

$$8\pi GM = \mu + \sum_{n=odd} \left(n! \partial_{i\mu}^{(n)} A(r, t)_{\mu=0} \right) \sin \mu . \tag{52}$$

If we define the temperature for an observer taking the time coordinate as t , then such an observer will define its imaginary time coordinate as μ . In such a case, then the surface gravity (temperature) perceived by the observer will be given by Equation (48), with μ defined as the solution to the equation

$$y = 8\pi GM - \sum_{n=odd} \left(n! \partial_{i\mu}^{(n)} A(r, t)_{\mu=0} \right) \sin \mu = \mu . \tag{53}$$

The solution for this equation is the intersection of the straight line $y = \mu$ and the function $y = 8\pi GM - \sum_{n=odd} \left(n! \partial_{i\mu}^{(n)} A(r, t)_{\mu=0} \right) \sin \mu$. It is then evident that the temperature perceived by an observer defining the time in agreement with t will differ from that defined by observers taking the time $T_0(r, t)$.

5. Conclusions

In this paper, we have calculated, by using two different methods, the black hole temperature for a spherically symmetric black hole in the non-linear formulation of massive gravity. For illustration, we have focused on the solution satisfying the relation $\beta = \alpha^2$. Then we reduce the theory to one free parameter. Interestingly, we have found that both methods, namely, the Path Integral method and the Bogolibov transformation one provides results consistent with each other. The results suggest that the observers defining the time coordinate in agreement with the Stückelberg function $T_0(r, t)$ will perceive a temperature equivalent to the one observed in GR. On the other hand, the observers defining the time arbitrarily t will perceive a different value of temperature with respect to the one perceived in GR with equivalent conditions of motion. This happens at large scales where the effects of the distortion of the time coordinate generated by the extra-degrees of freedom become appreciable. The distortion of the time coordinate affects the way how the observers define the coordinate t perceive the periodicity of the poles of the propagator if they use the Path Integral method in their calculations. Equivalently, from the perspective of the Bogoliubov method, the distortion of the time coordinate affects the relation between the advanced and the retarded time coordinates when we relate the vacuum over the future infinity (J^+) with the vacuum defined over the past null infinity (J^-) in the analytically extended Penrose

diagrams, which are descriptions of the process of gravitational collapse. This is the case because what the Penrose diagrams portrait is (1). The vacuum state before the formation of the black hole (J^-). (2). The vacuum state after the formation of the same black hole (J^+). The mentioned distortion of the time coordinate generated by the extra-degrees of freedom in Massive Gravity then affects the final result obtained for the coefficient $\beta_{\omega,\omega'}$, which is obtained after comparing the mentioned vacuum states. The same coefficient mixes the positive and negative frequencies. The effect described here is not standard, and it does not correspond to the emission of extra particles from the event horizon. At the event horizon scale, it is still expected that the black holes in Massive Gravity emit the same amount of particles as in GR. The effect discussed in this paper rather appears as a consequence of the ambiguity in the definitions of vacuum for the observers located at large scales with respect to the black hole. The definitions of vacuum, naturally, depend on how the time coordinate is defined for each observer.

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