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Review

Revenue Management in Airlines and External Factors Affecting Decisions: The Harmonic Oscillator Model

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Abstract: The Revenue Management (RM) problem in airlines for a fixed capacity, single resource and two classes has been solved before by using a standard formalism. In this paper we propose a model for RM by using the semi-classical approach of the Quantum Harmonic Oscillator. We then extend the model to include external factors affecting the people's decisions, particularly those where collective decisions emerge.

Keywords: revenue management; harmonic oscillator; collective decisions; spontaneous symmetry breaking

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1. Introduction

Revenue management (RM) is one of the most important concepts to be used inside a business or organization. It consists in maximizing the revenue from a fixed amount of perishable inventory by using techniques like "market segmentation" as well as "demand management" [1,2]. By using standard techniques, other authors have demonstrated that it is possible to increase the revenue in any organization if the market is segmented [1,2]. The market segmentation appears as a Revenue Function (RF) depending on the price (p_i) and the demand (d_i). In standard situations, the price and the demand are considered to be continuous stochastic variables [3]. In the case of airlines, an increase in ticket prices corresponds to a reduction in demand and vice versa. If we investigate in the literature a system with two variables, behaving in a way such that when one variable increases the other decreases and vice versa, we will find immediately that the harmonic oscillator performs well in this aspect. We can see the example of the pendulum, which for small oscillation angles follows the harmonic behavior [4]. For the pendulum, the largest amplitude of oscillation corresponds to a minimal velocity and vice versa, a maximal velocity corresponds to a minimal amplitude. Similar behavior can be found in several different systems, as far as they follow this harmonic behavior. Understanding that the harmonic oscillator behaves qualitatively as the RM system, we can then use it as a potential model for modelling the behavior of the price and demand. In this paper, we will consider the RM system as a semi-classical version of the Quantum Harmonic Oscillator (QHO) and then price and demand become operators related to each other at the quantum level through the uncertainty principle [5]. Remarkably, as it has been just mentioned, the Harmonic oscillator naturally reproduces some important features typically observed in the RM scenario in airlines; they are: (1) Higher price for one class means lower demand and vice versa. (2) A higher class, naturally being more expensive, has a lower demand, except when there is a promotion, such that the demand becomes much higher than for the case corresponding to the lower classes for the same price. (3) The different classes represent two dimensions of the classical phase space of the harmonic oscillator, and the highest possible revenue is proportional to the sum of the largest possible projected areas of the rectangles circumscribed inside the ellipses projected over the different planes of the ellipsoid representing the phase space covered by the harmonic oscillator. These three features make the harmonic oscillator scenario ideal for explaining the observed pattern of the RM problem in airlines. This also demonstrates how general the harmonic oscillator system is, given the wide range of applications it has [6]. The dynamics of this airline system is modelled by using a Hamiltonian function, with its corresponding free parameters. The basic RM model can be formulated with the simplest harmonic oscillator potential. However, when some external factors, such as the COVID-19, a hurricane or any other natural disaster enters in this scenario, then we have to introduce additional terms inside the potential function in the Hamiltonian of the oscillator, with their corresponding free parameters. In these situations, the purely harmonic behavior is completely lost. In this paper, besides illustrating how to implement the Quantum Harmonic Oscillator in the RM problem in airlines, we also illustrate one generic scenario where we model the collective decisions taken by a group of persons. The collective behavior scenario is modelled as a phase transition process emerging from the spontaneous symmetry breaking of a symmetry inside the system, in analogy to what happens in some scenarios in material science and particle physics [7,8]. In this way, once the symmetry of the vacuum (equilibrium condition) is broken, the system naturally selects an arbitrary condition which forces all the customers (related to the demand variable) to make the same decision, no matter what happens. The symmetry is in general broken due to some small fluctuation forcing the system to select a fixed vacuum state; this can only happen after the same system reaches certain conditions determined by its free parameters. We also analyze the consequences of the collective decisions on the total expected revenue, which will naturally differ from the standard situation. Note that the scenario illustrated in this paper is different to the one shown in [9], where standard probabilistic methods were mentioned. The paper is organized as follows: In Section 2, we formulate the standard RM problem in airlines and we explain in detail the two-class problem. In Section 3, we explain how the Quantum Harmonic Oscillator and particularly, its semi-classical approach, adapts naturally to the two-class problem. In Section 4, we explain how the collective decisions emerge by using the concepts of phase transitions due to the spontaneous symmetry breaking of a symmetry of the system. We also calculate the largest possible revenue when the vacuum symmetry is broken, which is naturally different from the standard case because the phase space volume covered by the system collapses dramatically when collective decisions are taken. We then compare the results emerging from collective decisions with those corresponding to the standard Harmonic oscillator case. In Section 5, we show a practical example where the formalism proposed here can be used, in order to analyze the revenue of the partnership between the airlines Hong Kong Express and Cathay Pacific. Finally, In Section 6, we conclude.

2. The Standard Revenue Management Problem in Airlines

The standard RM problem identifies different classes $i = 1, 2, \dots, n$ with the corresponding prices $p_1 > p_2 > \cdots > p_n$ and demands defined in an analogue way. The demand is in general a stochastic variable and the revenue management objective function is defined as

$$max\Pi^s = \sum_{i=1}^{n} p_i d_i(p_i), \tag{1}$$

with the fixed capacity of the plane constrained as follows:

$$\sum_{i=1}^{n} p_i d_i(p_i) \leq P \times C, \tag{2}$$

with C representing the capacity of the plane and P representing an average price. This condition just suggests that the revenue cannot overpass the plane's capacity. Market studies have demonstrated that a good market segmentation can increase the total revenue [1,2]. When the single resource (round trip) problem is analyzed, and considering only two classes, the first class tickets are priority in the sense that the company in charge is allowed to sell as many first class tickets as possible. However, in the practical sense, we understand that the demand for the first class is not able to fill the whole plane. Then, the mathematical problem connected with RM is to know how many second class tickets can be sold before starting to lose money for overselling them.

The Two Class Problem

Here we consider the demand for each class as X_1 (class 1) and X_2 (class 2), with the corresponding prices p_1 and p_2. Both demand variables are considered to be independent and continuous, with a probability density $f_{X_1}(.)$ and $f_{X_2}(.)$. As it was just mentioned, we will sell S_2 tickets for the second class and the problem consists in knowing when the selling process for the second class should be stopped such that we start to sell the tickets for the first class. Each class has the booking limits S_1 and S_2. Here $S_1 = C$ means that the first class (ideally) could be sold until filling the whole plane, something which certainly never happens. Given S_2, the expected profit is defined as

$$E[\Pi(S_2)] = p_2 S_2 + p_1 E_{min}(X_1, C - S_2). \tag{3}$$

This expression can be expressed more explicitly in the form

$$E[\Pi(S_2)] = p_2 S_2 + p_1 \left(E[X_1] + \int_{C-S_2}^{\infty} (C - S_2 - x_1) f_{X_1}(x_1) dx_1 \right). \tag{4}$$

Here, the first term is the revenue obtained from the second class. Additionally, $E[X_1]$ is the expectation value and $f_{X_1}(x_1)$ is the probability density distribution corresponding to the demand variable X_1. It has been demonstrated before that the maximal revenue is obtained when

$$F_{X_1}(C - S_2^*) = \frac{p_1 - p_2}{p_1}, \tag{5}$$

with F_{X_1} representing a cumulative density function of X_1, defined in the standard form as $F_{X_1}(C - S_2^*) = \int_0^{C-S_2^*} f_{X_1}(x_1) dx_1$. From Equation (5) we can obtain the optimal point where the revenue is maximal. That point is defined as

$$p_2 = p_1 \bar{F}_{X_1}(C - S_2^*), \tag{6}$$

with $\bar{F}_{X_1}$ being defined as $\bar{F}_{X_1}(C - S_2^*) = 1 - F_{X_1}(C - S_2^*)$. In [9], this result was interpreted as a limit value, after which we should stop selling tickets to the second class. Then, we can sell class 2 tickets as long as the ticket price is greater than or equal to the expected marginal seat revenue (EMSR) for the class 1. In general, if $S_2 < S_2^*$, then $p_2 > EMSR_1(C - S_2)$.

3. The Quantum Harmonic Oscillator for the Single Resource Two-Class Problem

A large demand in seats for some specific class corresponds to a low price. In the same way, a low demand for a particular class, corresponds to a high price of the same class. If we promote the price and the demand to be operators, then we can define the demand operator as $\hat{X}_i$ and the price operator as $\hat{p}_i$. A Quantum Mechanical system behaving in the same way as the price–demand relation for the RM problem in airlines is the Quantum Harmonic Oscillator (QHO) [6], although in this paper we will use its semi-classical version. If we consider first the quantum version of this system, then the operators corresponding to the price and the demand can be formulated such that they satisfy the standard commutation relations defined as

$$[\hat{p}_i, \hat{X}_j] = i\delta_{ij}, \qquad [\hat{p}_i, \hat{p}_j] = [\hat{X}_i, \hat{X}_j] = 0. \tag{7}$$

Then, when we explore the RM problem from the perspective of Quantum Mechanics (QM), we have to select which space we want to use for analyzing the system. This means that we can work either in the demand space or in the price space. They are both related to each other by the standard Fourier transformation defined as

$$\psi(p_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip_i x_i} \psi(x_i) dx_i. \quad (8)$$

For the customers, it is convenient to select the price space for analysis, because they are naturally interested only in the prices of the tickets. However, for the airline companies, since they are more concerned about the demand for the products which they offer, the demand space is the most convenient one for analyzing the system. Following the standard language of QM, we can interpret the price operator as the generator of the changes in the demand and vice versa; the demand operator for a product can be interpreted as the generator of the changes in prices. This is mathematically defined as

$$|x_i - a >= e^{-ip_i a/\hbar}|x_i >, \qquad |p_i + z >= e^{ix_i a/\hbar}|p_i > . \quad (9)$$

Now we can proceed to analyze the most basic situation. For practical calculations, however, we will consider the semi-classical approach to the harmonic oscillator problem.

The Two-Dimensional Case: Single Resource Two-Class Problem

If we analyze the two-dimensional version of the Quantum Harmonic Oscillator from the perspective of the RM problem in airlines, then each class corresponds to one dimension for the system. The Hamiltonian of the system is then defined as

$$\hat{H} = \sum_{i=1}^{2} \left(\frac{\hat{p}_i^2}{2} + \frac{1}{2}\omega_i^2 \hat{X}_i^2 \right). \quad (10)$$

The expected profit from the perspective of the Quantum Harmonic formulation can be derived in analogy with the result obtained in Equation (3). Evidently, instead of stochastic variables, the quantum formulation works with operators. However, for the analysis of the RM problem, the semi-classical approach of the oscillator is superior because the problem can be visualized graphically through the phase space. The Hamiltonian (10) has naturally two eigenvalues $E_{1,2}$, corresponding to each class. Since the first class is more expensive than the second class (for the same demand), then naturally it is expected that $E_1 > E_2$ in general. The Hamiltonian (10) reflects the typical behavior of the RM problem. Plotted in the phase space, the Hamiltonian for the two-class problem is a four-dimensional ellipsoid. As an illustration, we can observe Figure 1 [10]. Note that this figure illustrates a three-dimensional surface for the purpose of illustration; however, we have to take into account that here we are considering a four-dimensional ellipsoid.

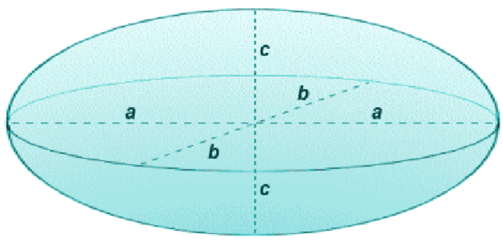

Figure 1. Ellipsoid illustrating the phase space covered by the Quantum Harmonic Oscillator. Without loss of generality, we can take $a = p_{max\,1}$ as the largest possible price for the first class and $b = X_{max\,1}$ as the largest possible demand for the first class. $c = p_{max\,2}$ might represent, for example, the largest possible price for the second class, with the corresponding largest demand for the second class given by a_{max}. Although we are talking about a four-dimensional ellipsoid, this figure illustrates a three-dimensional example.

Since $E_1 > E_2$, then the projected area of the two-dimensional ellipsoid corresponding to the first class is larger than the projected area corresponding to the second class. In general, the more expensive a class is, the taller will be the height of the projected rectangle. The harmonic oscillator model is so perfect for being adapted in the RM case that the phase space in Figure 1 illustrates the most interesting behavior related to this model, namely, in the hypothetical case when the first class is cheaper than second class, which can certainly happen under some short-time promotion or so, the demand becomes much larger than the one corresponding to the second class. This is something which certainly happens the whole time in reality. For example, at a supermarket during a day when expensive products are offered at a cheaper price, you will naturally see a huge amount of people, more than the usual, buying the most expensive products and probably ignoring the cheapest ones. This only happens because the customers will not see in the short term another chance of obtaining such expensive products at prices lower than the cheapest products. The same principle applies to the RM problem in airlines. As we have mentioned before, the Hamiltonian (10) obeys the eigenvalue problem

$$\hat{H}\psi(x_i,t) = E_i\psi(x_i,t). \tag{11}$$

The ellipsoid equation emerging from the semi-classical scenario is

$$1 = \frac{1}{2E}\omega^2 X^2 + \frac{1}{2E}p^2, \tag{12}$$

with $p^2 = p_1^2 + p_2^2$ and $X^2 = X_1^2 + X_2^2$. The most important quantity found by then was the largest possible revenue, which from the semi-classical perspective becomes a geometrical optimization problem for finding the largest possible rectangle which can be circumscribed inside the projected ellipse for each class. The projected ellipses are defined by the standard equation

$$1 = \frac{1}{2E_{1,2}}\omega^2 X_{1,2}^2 + \frac{1}{2E_{1,2}}p_{1,2}^2. \tag{13}$$

The optimization result is

$$E[\Pi] = A_1 + A_2 = \frac{2E_1^2}{\omega_1^2} + \frac{2E_2^2}{\omega_2^2}, \tag{14}$$

which for the case of an isotropic harmonic oscillator, becomes

$$E[\Pi] = \frac{2}{\omega^2}\left(E_1^2 + E_2^2\right). \tag{15}$$

If we use the standard expression for the eigenvalues of the QHO, we have

$$E_{1,2} = \omega\left(n_{1,2} + \frac{1}{2}\right). \tag{16}$$

Replacing this expression in Equation (15), we obtain

$$E[\Pi] = 2\omega\left(\left(n_1 + \frac{1}{2}\right)^2 + \left(n_2 + \frac{1}{2}\right)^2\right). \tag{17}$$

The total "energy" of the system $E = E_1 + E_2$, or more explicitly

$$E = \omega(n_1 + n_2 + 1), \tag{18}$$

is the constraint which complements (15). Equation (18) is just a way to express the values taken by the Hamiltonians of each class, $\hat{H}_1$ and $\hat{H}_2$. Although we could analyze practical examples by using the numbers n_1 and n_2, we will still use the standard Hamiltonian and its phase space in order to analyze practical problems. The number n_i just fixes the value of the Hamiltonian. Then, we can write, for example,

$$\hat{H} = \omega(n_1 + 1/2) = \frac{1}{2}\omega^2 X^2 + \frac{1}{2}p^2, \tag{19}$$

for one class. Then, n_1 just fixes the value taken by $\hat{H} = E$. Please see the Section 5 for the practical applications of the present formalism.

4. Collective Decisions in the RM Problem: Spontaneous Symmetry Breaking

We can extend the QHO model for including additional terms in the potential. These additional terms are related to the decisions taken by the passengers, influenced by some external factors. Under some special circumstances, these decisions turn out to be collective. Collective decisions can be interpreted as spontaneous symmetry breaking of the system if we consider the price and the demand as Quantum Fields. Similar situations emerge in finance and other research areas where collective decisions emerge naturally [11,12]. Here we will analyze the standard Hamiltonian of the form

$$\hat{H} = \sum_i \left(\frac{\hat{p}_i^2}{2m} + \frac{1}{2}\omega^2 \hat{X}_i^2 + \lambda \frac{\hat{X}_i^4}{4} \right), \tag{20}$$

where we can perceive that the potential term has the form

$$V(\hat{X}) = \frac{1}{2}\omega^2 \hat{X}^2 + \lambda \frac{\hat{X}^4}{4}. \tag{21}$$

This is a well-understood potential term in the research area of Quantum Field Theory [7,8]. In this part of the paper, we will not take into account the different classes because when the symmetry is broken, the distinction between classes is completely lost. Then, we will omit the subindex i from Equation (20) for the subsequent analysis. The potential (21) is a function where some combinations of the parameters ω and λ create the ideal scenario for reproducing spontaneous symmetry breaking after a small fluctuation affects the system [13–19]. Here we will interpret this symmetry breaking pattern in the language of the RM problem. We will also understand how the phase space is deformed due to the presence of the additional potential term $\lambda \frac{\hat{X}^4}{4!}$. The form of the potential (21) can be illustrated in Figure 2.

The figure shows that the spontaneous symmetry breaking occurs when $\omega^2 < 0$, such that the system develops more than one vacuum condition. If the demand operator has infinite components, then the system would develop infinite vacuum states, all of them degenerate. In the thermodynamic limit, any small fluctuation will force the system to select one of these vacuum states arbitrarily, defining then the collective decision pattern taken by all the persons affected by some particular event. The vacuum states appearing in Figure 2, can be obtained mathematically from Equation (21) if we calculate the condition $\partial V/\partial x = 0$, obtaining then the vacuum conditions $x_0^2 = \omega^2/\lambda$, valid when $\omega^2 < 0$. Here we understand that $x = \hat{X}\psi(x,t)$ and we take it as a Quantum Field. Note from the figure that when the symmetry is spontaneously broken, the usual vacuum state with $x_0 = 0$ becomes unstable and then the most stable condition turns out to be the one corresponding to one of the possible states breaking the symmetry spontaneously. Note that in the thermodynamic limit, a small fluctuation makes the system select from one of the degenerate states. Such fluctuation in the practical sense, and for our purposes in this paper, corresponds to the rumor propagation due to some catastrophic event. Catastrophic events like bad weather, volcanic explosions blocking air traffic, or even a pandemic disease affecting a region or the world, just create the ideal scenario for the standard vacuum state to become unstable. This means that the information of the catastrophic events is stored inside the free parameters of the potential defined in Equation (21). In the light of the

COVID-19 crisis, perhaps the worst pandemic crisis in this century [20], this kind of analysis became important. Due to the COVID-19 pandemic, for example, all the airlines had to stop most of the services. The demand for tickets then became low and the corresponding prices became too high, as it could be verified at the time by any airline website. Low-fare airlines stopped services completely and big airlines were only providing partial services. In this way, the offer is limited because the demand fell catastrophically due to the quarantine measures, social distance measures and any other measures affecting travel taken by different countries. By keeping inside the harmonic oscillator model, we would say that the phase space has been reduced considerably due to a catastrophic event. Spontaneous symmetry breaking is the most natural explanation to what was happening during the COVID-19 period. Basically, the pandemic itself (together with all the measures) created the symmetry-breaking scenario by changing the value of the free parameters in the system, provoking then the demand field to be fixed at some small value defined by some specific vacuum state ($x_0 = \omega/\lambda^{1/2}$ if we only take the positive branch of the solution). Note that the new Hamitonian (20) defines a phase space in agreement with Figure 3.

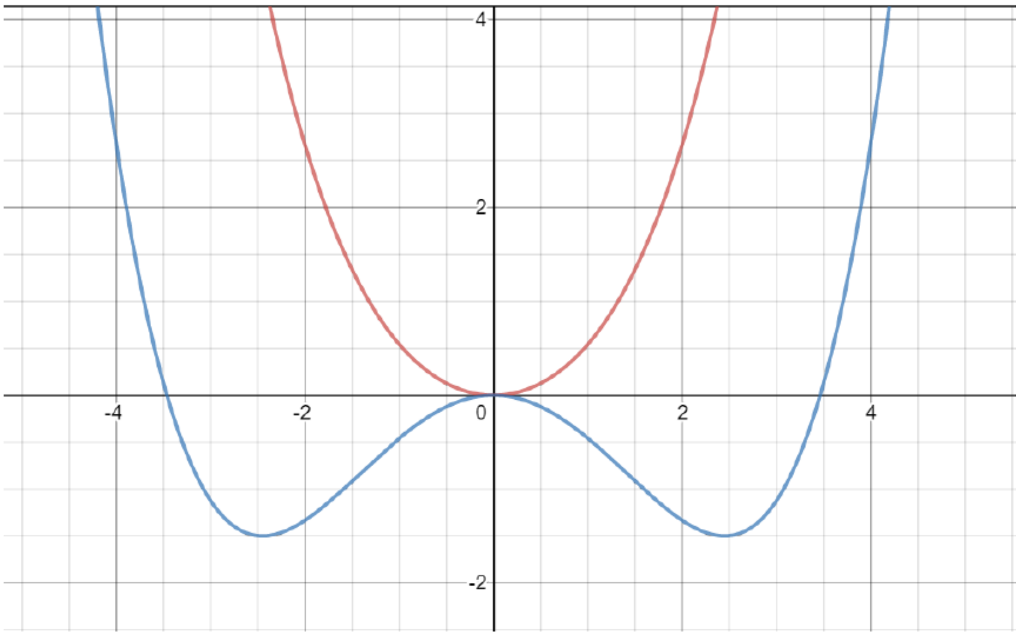

Figure 2. Symmetry breaking pattern with the x-axis representing the demand and the y-axis representing the price. The red line corresponds to the standard situation where both ω^2 and λ are positive. In this case, we only have one well-defined vacuum state. On the other hand, the blue curve corresponds to the case where $\omega^2 < 0$. This situation forces the system to develop two (or infinite) different vacuums as illustrated in the figure. The most stable state when $\omega^2 < 0$ corresponds to the one where the symmetry is spontaneously broken.

Note in addition that although the phase space looks larger for the new Hamiltonian (20), in comparison with the standard Harmonic oscillator case (10), the total expected revenue falls considerably. The total phase space volume is naturally larger when the symmetry is spontaneously broken because the larger possible prices also become catastrophically high because the airline companies cannot exploit fully the whole available phase space. This is because each portion of it now lives in a different Hilbert space, disjoint from each other in the thermodynamic limit [13,21]. Then, although the plot suggests larger available phase space, the small fluctuation provoked by the border measures due to the pandemic or any other disaster makes the system focus only on one single line of the phase space. Calculating the total expected revenue is not a problem in this case where there is no distinction between first or second class. By taking $x_0^2 = \omega^2/\lambda$ (with $\omega^2 < 0$), and if we replace this result in Equation (20), after taking

$$E = \frac{p^2}{2m} - \frac{1}{2}\omega^2 x^2 + \lambda \frac{x^4}{4}, \tag{22}$$

as the result of $\hat{H}\psi(x,t)$, we obtain the corresponding value for the price defined on the symmetry breaking vacuum state as

$$p_0 = \sqrt{2mE + \frac{m\omega^4}{2\lambda}}. \tag{23}$$

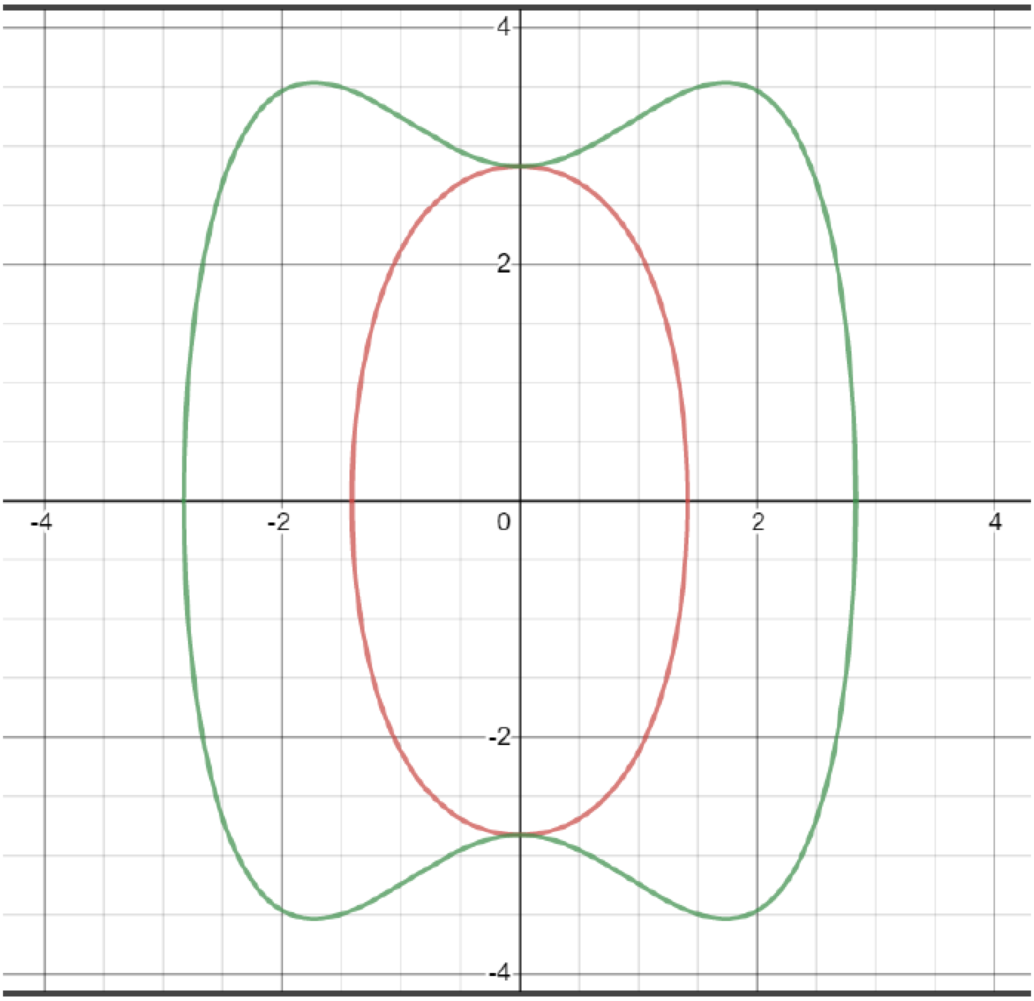

Figure 3. Changes on the phase space due to the presence of the quartic-order term in the Hamiltonian (20). The x-axis represents the demand and the y-axis represents the price. Note that the available phase space looks bigger when the quartic term appears. However, due to spontaneous symmetry breaking, the phase space is not fully available and it collapses to a line.

Here E is assumed to be fixed for each class. However, under spontaneous symmetry breaking, there is no distinction between classes and for that reason we will not be concerned about this classification. The result (23) represents the price of the ticket when the system has selected a particular vacuum state. Then, the expected revenue under spontaneous symmetry breaking (SSB) would be

$$E[\Pi]_{SSB} = p_0 x_0 = \omega \left(\frac{m}{\lambda}\right)^{1/2} \sqrt{2E + \frac{\omega^4}{2\lambda}}. \tag{24}$$

Then, for a fixed E, the expected revenue depends on the parameters m, ω and λ. By taking the semi-classical results already defined in Equation (18), we obtain

$$E[\Pi]_{SSB} = \omega^{3/2}\left(\frac{m}{\lambda}\right)^{1/2}\sqrt{(2n+1)+\frac{\omega^3}{2\lambda}}, \tag{25}$$

where we have taken the positive branch of the product between the price and the demand and we are considering only one class because the distinction already disappears at this level. If we compare this result with the expected revenue obtained from the harmonic oscillator model in Equation (17), by considering only one class, and considering $n = 0$, we obtain the condition such that the largest expected revenue in the standard harmonic oscillator system defined in Equation (10) is larger than the expected revenue in the present case. Such condition is defined by

$$\frac{\lambda^2}{2m\omega} \geq 2\lambda + \omega^3. \tag{26}$$

This condition is naturally satisfied for small frequencies, which typically happens in these situations. We can solve Equation (26) for λ, obtaining then

$$\lambda \geq 2m\omega\left(1+\sqrt{1+\frac{\omega^2}{2m}}\right), \tag{27}$$

which for small frequencies becomes approximately $\lambda \geq 4m\omega$. In this way, we have demonstrated that the expected revenue during crisis periods where the symmetry is spontaneously broken due to small external fluctuations, after the system reaches certain conditions, is much lower than the expected revenue under standard circumstances in most of the situations. We have also identified the parameters responsible of this behavior. The most important parameter is evidently the frequency of the system ω, which not only controls the vacuum behavior and whether we have symmetry breaking or not, but in addition, its absolute value marks the limits where the expected revenue falls with respect to the standard case. Note that we can take ω as the usual frequency, ignoring the fact that $\omega^2 < 0$ because we have already made the corresponding change of sign in Equation (22).

5. Numerical Example

In this section, we will work over some numerical examples for testing our model. We must remark that the QHO is a simplification of a general tendency. For the simplest case, the most general form for the Hamiltonian explaining the RM problem in airlines is

$$\hat{H} = a_0\hat{p} + b_0\hat{x} + a_1\hat{p}^2 + b_1\hat{x}^2 + a_2\hat{p}^3 + b_2\hat{x}^3 + a_3\hat{p}^4 + b_3\hat{x}^4 + \ldots \tag{28}$$

The degree of the polynomial expansion for the price and demand changes for different cases. For simplicity, in this article we have used the standard harmonic oscillator as a starting point for explaining the dynamical relation between price and demand in the airlines. In what follows, we will illustrate how the formulation proposed in this paper can be applied to a real-life situation. We will use as example the airline Hong Kong Express. The authors checked the website for Hong Kong Express on Thursday 15 February, and found that the lowest price for the tickets for going from Hong Kong to Tokyo was HKD 580. The same price repeated for several days and there were fluctuations over this price, with a peak in price of HKD 2888 on Sunday 25 February. In a non-hot season, a pattern of having more expensive tickets on Fridays was constant, with a price of HKD 608. The hot season by the end of March showed a tendency to have very high prices. The following table shows the values for the prices for different dates corresponding to the period between 16 February and 31 March. Based on the simplest model with the semi-classical approach of the Quantum Harmonic Oscillator, we made the predictions for the demand values. Although knowing the exact demand is impossible (it must be always conjectured), the model shows consistency in the sense that by controlling the price, the airlines can control the demand. In other words, in hot seasons, understanding that more people are willing to travel, the airlines increase the prices of the tickets in order to reduce the demand. We can then say that the airlines operate over the price space, and are able to control this variable. On the contrary, the customers operate over the demand space, because they have the power of deciding whether or not they buy a ticket.

Since Hong Kong Express is a cheap airline, considered to be a class-less airline in the sense that there is only one class, for this system we have a two-dimensional harmonic oscillator behavior. If we want to extend this system to include one more class, we have to take Hong Kong Express as a second class section and its partner airline, namely, Cathay Pacific, would correspond to the first class. In other words, we will take the economy class of Cathay Pacific as the first class for Hong Kong Express. This analysis is valid since both airlines correspond to the same company. We could notice that the price for the same mentioned period for Cathay Pacific represents a constant price for the economy class of HKD 5482. The reason for this price not to change is that the demand for this ticket is always low no matter the season; in other words, there is never enough demand for the airline (Cathay Pacific) to force it to decrease by increasing the price. Additionally, the proposed price is the minimal possible price for this airline. Evidently, most of the customers will select Hong Kong Express for this itinerary and this explains why the prices fluctuate season by season for this case. The explicit Hamiltonian for this system is

$$\hat{H} = a_0\hat{p}_{CX}^2 + b_0\hat{x}_{CX}^2 + a_1\hat{p}_{HKE}^2 + b_1\hat{x}_{HKE}^2. \tag{29}$$

This Hamiltonian is just a special case for the most general situation described in (28). It can be expressed in a compact way as

$$\hat{H} = \hat{H}_{CX} + \hat{H}_{HKE}. \tag{30}$$

We can define the parameters for each Hamiltonian $\hat{H}_{CX}$ and $\hat{H}_{HKE}$ separately. Ideally, in a single day, when the estimated demand increases so much, the airline then increases the price of the ticket until the demand decreases to a value equal to the plane capacity C (number of available seats). Let us define C_{CX} and C_{HKE} as the plane capacities for the corresponding airlines. Then, for example, the price HKD 2888 in the Table 1 corresponds to $x_{HKE} = C_{HKE}$. Then, we have the following information for the case of February 25th for Hong Kong Express:

$$\begin{aligned} \hat{H}_{HKE} &= (236)^2 + a_1(2,888)^2, \\ \hat{H}_{HKE} &= (4 \times 236)^2 + a_1(588)^2. \end{aligned} \tag{31}$$

In this previous expression, we have set $b_1 = 1$ in Equation (29) and we have left a_1 to be fixed by the initial conditions established in Equation (31). After solving the system, we find that $a_1 = 47.8$ with the corresponding units. Note that in Equation (31) we have fixed the demand at the largest price p_{HKE} = 2888 HKD with the plane capacity. After investigating on the websites, we found that Hong Kong Express' largest plane is an Airbus A321neo with a capacity of $C_{HKE} = 236$ passengers [22]. Then, we assume that the airline is using its largest plane for this route. Finally, if the price offered to the market is p_{HKE} = HKD 588, namely, the lowest price found during ordinary seasons, then we can assume that the demand related to this price in a day like 25 February would be four times the plane capacity, namely $x_{HKE} = 4 \times 236$ passengers for the lowest price. This is the information contained in Equation (31). Here we have two assumptions and they are: (1) The demand is four times the capacity of the plane when the price is the lowest during a hot day. (2) The demand is just equal to the plane capacity when the airline fixes the price to a largest possible value. In this way, we can say that on February 25th, the Hamiltonian for Hong Kong Express is

$$\hat{H}_{HKE} = 47.8p_{HKE}^2 + x_{HKE}^2. \tag{32}$$

For CX (Cathay Pacific), we have to fix the Hamiltonian in a similar way. However, we did not detect any fluctuation in the prices for the lowest fare of CX, which was fixed to $p_{CX} = 5482$, which is also the lowest possible fare. This fare is also special because it offers a free return ticket if the round trip option is selected. Yet HKE (Hong Kong Express) is still cheaper if a round trip is selected. The only advantage of CX is the free food service plus the commodity of the seats. However, the itinerary Hong Kong–Tokyo is only a 4 h trip more or less and many people decide not to take any meals during this flight when they use HKE. This makes us think that the demand for the CX tickets is so low that the plane is probably filled with a quarter or lower of its capacity. Then, we will assume $x_{CX} = \frac{1}{4}C_{CX}$. The most expensive ticket for CX during the same itinerary costs around HKD 9500. For this particular case, this price would correspond to a zero value for the demand, namely $x_{CX} \approx 0$ when p_{CX} = HKD 9500. Then, the conditions for the Hamiltonian of CX are

$$\begin{aligned}\hat{H}_{CX} &= a_0(9500)^2,\\ \hat{H}_{CX} &= a_0(5482)^2 + b_0(202/4)^2.\end{aligned} \tag{33}$$

Here we are taking $C_{CX} = 202$ passengers, considering that the smallest plane of CX is the AirbusA321neo [23]. Here again, considering $b_0 = 1$, we obtain $a_0 = 4.23 \times 10^{-5}$. Then, the Hamiltonian for 25 February for CX is

$$\hat{H}_{CX} = 4.25 \times 10^{-5} p_{CX}^2 + x_{CX}^2. \tag{34}$$

Then, the Hamiltonian (29) is defined as

$$\hat{H} = 4.25 \times 10^{-5} p_{CX}^2 + x_{CX}^2 + 47.8 p_{HKE}^2 + x_{HKE}^2. \tag{35}$$

We can perceive the difference in price and demand scales, for this itinerary, between CX and HKE. The differences are reflected through the parameters b_0 and b_1, which differ from each other by six orders of magnitude. With the Hamiltonian (35), CX and HKE in partnership can decide if it is necessary to increase or decrease the prices for certain itinerary. Note that the Hamiltonian (35) is valid for one day. However, by changing the parameters b_0 and b_1, we can adapt the same Hamiltonian to different days. In other words, the parameters b_0 and b_1 change day by day in agreement with the maximum and minimum prices fixed by the airlines. Finally, we must remark that the quadratic dependence of the prices and demands for $\hat{H}_{CX}$ and $\hat{H}_{HKE}$ are shown for illustration purposes. However, for more general cases, the dependence might change. Yet still, the quadratic dependence represents a good initial ansatz. Our approach is just limited for the lack of demand date, something which is not provided by the airlines. Finally, we can evaluate the revenue for CX and HKE if we know the values of $\hat{H}_{CX}$ and $\hat{H}_{HKE}$. These values can be found by evaluating any of the pairs of price–demand values mentioned previously. The result is $\hat{H}_{CX} = 3.84 \times 10^3$ and $\hat{H}_{HKE} = 3.99 \times 10^8$. The basic optimization helping us to find the largest possible area of a square circumscribed inside an ellipse gives us the largest possible revenues for CX and HKE for the analyzed itinerary as

$$\begin{aligned}E[\Pi]_{CX} &= 9.25 \times 10^5 \text{ HKD},\\ E[\Pi]_{HKE} &= 2.75 \times 10^7 \text{ HKD},\end{aligned} \tag{36}$$

where we have divided the optimal result by four in order to take into account that the price and the demand are both positive numbers. The result (36) means that for the same itinerary and the same date, the cheapest fares offered by HKE give higher profits than the expensive fares offered by CX. This is a demonstration of the importance of segmentation and the emphasis on massification at the moment of offering a product. If there is a pandemic, only the expensive airlines would operate (and this was actually the case during COVID-19), and this represents significant losses, considering that HKE gives two order of magnitude more revenue than CX in the present example. During the COVID-19 pandemic, the demand was so low that in such a case, only the price p_{CX} = HKD 5482 would be available, no matter the demand. This is the case because the demand will be always much smaller than the plane's capacity. This catastrophic scenario is what generated a huge crisis in the airline industry and it is equivalent to a phase transition of the system, looking for a new ground state. This was the scenario illustrated in the Section 4.

Table 1. Special values for the prices for Hong Kong Express. The itinerary corresponds to the trip Hong Kong–Tokyo. The period under analysis was 15 February 2024–5 April 2024. All the prices during this period are equal to HKD 588, except for most of the dates appearing in the table.

Key Dates *P*	Price HKE (HKD)	Day of the Week
16 February	608	Friday
23 February	708	Friday
24 February	1508	Saturday
25 February	2888	Sunday
1 March	608	Friday
3 March	658	Sunday
4 March	658	Monday
5 March	658	Tuesday
8 March	608	Friday
9 March	652	Saturday
10 March	**588**	Sunday
14 March	658	Thursday
15 March	708	Friday
20 March	728	Wednesday
21 March	1068	Thursday
22 March	938	Friday
23 March	918	Saturday
24 March	918	Sunday
25 March	788	Monday
26 March	1268	Tuesday
27 March	1488	Wednesday
28 March	2018	Thursday
29 March	1588	Friday
30 March	1318	Saturday
31 March st	818	Sunday
5 April	608	Friday

Comparison with Other Methods

The standard methods used for analyzing the RM problem in airlines are based on the formalism illustrated in the Section 2. In such a case, the relation (2) already indicates that a large price for a ticket is related to a low demand and vice versa. The expected revenue is indicated by Equation (4) and the maximum revenue condition is indicated by Equation (5). However, the problem with the traditional method proposed in Section 2 is that the maximum revenue condition does not indicate an explicit relation between price and demand as can be observed from Equation (5); instead, this dependence appears via a probability density function $f_{X_1}(x_1)$, which must be pre-established from previous data analysis. This necessity of knowing the exact density distribution of the data is a significant disadvantage with respect to the harmonic oscillator method, which avoids this and other complications by providing analytic dependencies between price and demand via the phase space through the Hamiltonian formulation. Then, our method gives an initial good estimation of the price–demand relation for airlines without invoking statistical methods. Subsequently, the statistical methods are just used for adjusting the free parameters of the theory, here taken as those appearing in Equation (28). In other words, our method suggests in advance a functional dependence between price and demand, leaving the statistical methods just for refining the values of the free parameters involved in the analysis. Then, even without involving any statistical method, the proposed formalism gives us a good qualitative behavior of the price–demand relation when different classes are involved. Additionally, the related optimization problem, illustrated in Equation (14) and used for calculating the maximum revenue, is just a standard optimization problem from ordinary calculus, which is much simpler than the way the standard methods are able to calculate the ideal revenue via statistical probability density function. In summary, our method is mainly analytical and it just uses statistics as an auxiliary tool, while the other methods present a full statistical dependence.

6. Conclusions

In this paper we have developed an RM model based on an extension of the harmonic oscillator case. We have included a quartic-order term which is related to the interaction between consumers taking decisions based on external events. If $\omega^2 > 0$, then we have the usual standard harmonic oscillator case. This means that the quartic-order term does not affect the harmonic behavior so much in such a case. However, when $\omega^2 < 0$, the conditions of the system change completely. In such a case, what is considered to be the most stable condition becomes unstable. Such instability can be due to a catastrophic event, such as the current COVID-19 pandemic situation, for example. Then, a small fluctuation (rumors of people, news or information) of the system, forces it to select some vacuum state, which corresponds to a fixed price and demand. In this way, the whole phase space collapses to a single line. The product of the price and the demand, evaluated both on the symmetry breaking vacuum state, gives us the expected revenue for this case. We have demonstrated in Equation (26) that the expected revenue for the case where the symmetry is broken is lower than for the case where we have the standard behavior based on the semi-classical approach of the Quantum Harmonic Oscillator (semi-classical model). This is valid when the frequency ω is small under the standard units taken in the system. Here, evidently, we assume the same frequency in both cases (the standard case and the symmetry breaking case) for the purposes of comparison. Finally, we have illustrated an example where the Hamiltonian approach can be used for analyzing the price and demand for the partnership between CX (Cathay Pacific) and HKE (Hong Kong Express). We have proved that the segmentation of the airline's fare by including a cheaper option, generating massification, is the key ingredient for increasing the revenue of an airline company. Our approach also shows how to deal with the cases where there are partnerships between organizations. The proposed method has also a significant advantage with respect to the traditional way of calculating the RM in airlines, because it does not depend explicitly on prior statistical analysis. Instead, the statistics are reduced to the analysis of some free parameters which can be adjusted with the observations correspondingly.

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