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A Possible Solution to the Black Hole Information Paradox

Ivan Arraut

Institute of Science and Environment and FBL, University of Saint Joseph, Estrada Marginal da Ilha Verde, 14–17, Macao 999078, China; ivan.arraut@usj.edu.mo

Abstract: The information paradox suggests that the black hole loses information when it emits radiation. In this way, the spectrum of radiation corresponds to a mixed (non-pure) quantum state even if the internal state generating the black hole is expected to be pure in essence. In this paper we propose an argument solving this paradox by developing an understanding of the process by which spontaneous symmetry breaks when a black hole selects one of the many possible ground states and emits radiation as a consequence of it. Here, the particle operator number is the order parameter. This mechanism explains the connection between the density matrix, corresponding to the pure state describing the black hole state, and the density matrix describing the spectrum of radiation (mixed quantum state). From this perspective, we can recover black hole information from the superposition principle, applied to the different possible order parameters (particle number operators).

Keywords: information paradox; black holes; Hawking radiation; spontaneous symmetry breaking

PACS: 03.65.−w Quantum mechanics; 03.67.−a Quantum information; 04.62.+v Quantum fields in curved spacetime; 04.70.Dy Quantum aspects of black holes, evaporation, thermodynamics

1. Introduction

Black holes are compact objects containing huge amounts of mass in a very small space-time region [1,2]. They develop a surface called the event horizon. Once a particle enters the event horizon, classically, it can never escape, no matter how much energy is invested in the process. Initially, it was believed that black holes were completely black, unable to emit particles. However, it was subsequently demonstrated by Hawking that the black holes also emit radiation [3,4]. This result was important because without this emission of radiation, some fundamental laws of thermodynamics would be violated [5–7]. For example, the first law of thermodynamics requires an appropriate definition of temperature in order to respect energy conservation. The Hawking radiation, however, brought, by itself, one problem: namely, the famous information paradox [8]. The black hole information paradox had been formulated by Hawking by the time he discovered the black hole evaporation process. Hawking realized that while the radiation emerging from a black hole is thermal in nature, and it only depends on the mass, charge, and angular momentum of the black hole, there are still an infinite number of ways to generate the same black hole with the same macroscopic properties. Thus, the same thermal radiation coming from a black hole, could be developed by any of the infinite possible microstates (internal states) consistent with the black hole macrostate with mass M, charge Q and angular momentum J. While each internal configuration of the black hole represents a pure state, the thermal radiation, being associated with all possible internal configurations, is represented by a mixed quantum state. This is the case because there is no correlation between thermal radiation, and

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as such, information cannot be carried out from the black hole. This lack of correlation occurs because the backreaction of the emission is not being counted. This is basically the root of the information paradox. In the past, other authors have proposed potential solutions to the black hole information problem. In [9], a possible solution was found when a non-thermal spectrum was found after calculating the backreaction. Then, in [10], it was demonstrated that there are correlations in a non-thermal spectrum. It was also found in [11] that entropy is conserved during black hole evaporation. In this paper, we take a different point of view and we propose an argument for solving the paradox. The starting point is the fact that black hole evaporation can be expressed as a natural consequence of the spontaneous breaking of the symmetry under exchange of internal configurations [12]. This symmetry keeps the same mass M, angular momentum L, and charge Q invariant. From this perspective, the vacuum expectation value of the particle number, defined as $< 0|\hat{n}_i|0 >$, corresponds to an order parameter, being zero before the formation of the black hole and non-zero after its formation. When the black hole has some specific mass, angular momentum, and charge, it has infinite possible ground states. In other words, we have a vacuum degeneracy, typical from in processes involving spontaneous symmetry breaking. When the black hole selects one among the many possible ground states, it emits radiation, and this is equivalent to tracing out all the other possible vacuum states. The difference between this case and the ordinary breaking of spontaneous symmetry is that here, we have the possibility of having an entanglement between the different possible ground states. The information paradox disappears when we sum all the possible order parameters (particle number operators), showing that this sum must be equal to zero, and recovering, in this way, the trivial ground state before the gravitational collapse. Then, the original ground state, before the formation of the black hole, is equal to the sum of all the possible ground states after the formation of the black hole. We can interpret this as the possibility of emitting antiparticles in addition to the ordinary particles.

2. The Schwarzschild Solution

For simplicity, we will focus on the Schwarzschild case. For the other cases, the extension is direct. The metric is defined as [1]

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (1)$$

The event horizon is defined at $r_H = 2GM$. Classically, when a particle approaches a distance smaller than r_H, then it cannot escape from the gravitational influence of the black hole. However, if we use arguments of Quantum Mechanics, then some particles can escape, giving rise to a spectrum of thermal radiation. The arguments developed by Hawking are explained in this section.

The Black Hole Evaporation Process

The black hole evaporation process emerges from a comparison between the vacuum state before the formation of the black hole and the vacuum state after the formation of the same body [13]. Before the formation of the black hole, the vacuum state is trivial or devoid of particles, namely

$$\hat{b}_{\mathbf{p}}|\bar{0}>= 0. \quad (2)$$

This vacuum state corresponds to the field expansion

$$\phi(x,t) = \sum_{\mathbf{p}}\left(f_{\mathbf{p}}\hat{b}_{\mathbf{p}} + \bar{f}_{\mathbf{p}}\hat{b}^+_{\mathbf{p}}\right). \quad (3)$$

After the formation of the black hole, the vacuum becomes nontrivial, and it is now defined by

$$\hat{a}_{\mathbf{p}}|0>=0. \tag{4}$$

This vacuum state then corresponds to the field expansion

$$\phi(x,t)=\sum_{\mathbf{p}}\left(p_{\mathbf{p}}\hat{a}_{\mathbf{p}}+\bar{p}_{\mathbf{p}}\hat{a}^{+}_{\mathbf{p}}+q_{\mathbf{p}}\hat{c}_{\mathbf{p}}+\bar{q}_{\mathbf{p}}\hat{c}^{+}_{\mathbf{p}}\right). \tag{5}$$

It is important to remark that the field expansions in Equations (3) and (5) contain the same amount of information. The difference is with respect to which vacuum state we expand, and, as a consequence, with respect to which modes we are expanding the quantum field with. The effects of radiation emerge when we compare the vacuum states of Equations (2) and (4). This comparison is possible via Bogoliubov transformations, which are able to relate the quantum operators as

$$\hat{a}_{\mathbf{p}}=u_{\mathbf{p},\mathbf{p}'}\hat{b}_{\mathbf{p}'}-v_{\mathbf{p},\mathbf{p}'}\hat{b}^{+}_{\mathbf{p}'}. \tag{6}$$

Then, when we try to annihilate the ground state defined in Equation (2) with the operator $\hat{a}_{\mathbf{p}}$, Equation (6) suggests

$$<\bar{0}|\hat{n}^{a}_{\mathbf{p}}|\bar{0}>=|v_{\mathbf{p},\mathbf{p}'}|^{2}. \tag{7}$$

This means that the ground state now is full of particles, appearing through a spectrum of radiation. It has been proved before that

$$<\bar{0}|\hat{n}^{a}_{\mathbf{p}}|\bar{0}>=\frac{\Gamma_{\mathbf{p},\mathbf{p}'}}{e^{\frac{2\pi\omega}{\kappa}}\pm 1}. \tag{8}$$

The fact that the spectrum of radiation emerges from a non-zero value of the Bogoliubov coefficient $v_{\mathbf{p},\mathbf{p}'}$ means that the Hawking radiation emerges from the mix of positive and negative frequency modes.

3. The Formulation of the Information Paradox

The no-hair theorem of black holes suggests that the physical state of a black hole can be characterized by its mass M, angular momentum L, and charge Q [1,14]. These three independent parameters are consistent with the huge amount of possible internal states of a black hole Ω. Even still, if Hawking's calculation is right, then this means that the thermal spectrum is independent of the details regarding how the particles inside the black hole are arranged [13]. Thus, technically, the information about the internal details of the black hole is lost. Another way to perceive this is by understanding that while each internal configuration of the black hole corresponds to a pure quantum state, the spectrum of radiation corresponds to a mixed quantum state. A pure quantum state obeys a unitary evolution as follows

$$|\psi(t_1)>=U(t_1,t_2)|\psi(t_2)>, \tag{9}$$

and we can always express it as a ket (wave function). More generally, it is also possible to express the quantum state through a density matrix, as $\hat{\rho}=|\psi><\psi|$. For pure and mixed states, the trace of this operator is $Tr(\hat{\rho})=1$. However, although for pure states the idempotent condition $\hat{\rho}^2=\hat{\rho}$, or equivalently, $Tr(\hat{\rho}^2)=1$, for mixed states this condition is violated, and, in general, $Tr(\hat{\rho}^2) \leq 1$. Although there is no ket representation for mixed states, they can still be expressed with a density matrix of the form

$$\hat{\rho} = \sum_{k=1}^{N} p_k |\psi_k >< \psi_k|. \tag{10}$$

Here, $|\psi_k >$ is some set of pure states. Equation (10) is then a superposition of pure states. The essential idea behind the information paradox is that the black hole radiation is thermal, because it corresponds to a mixed quantum state, which is a superposition of the pure states represented by all the possible internal configuration arrangements of the black hole.

4. The Information Paradox from the Perspective of Spontaneous Symmetry Breaking

The black hole evaporation process can be expressed as a consequence of the mechanism of spontaneous symmetry breaking, where the black hole, having access to a huge amount of possible ground states, selects one arbitrarily. When this occurs, then the black hole emits particles in the form of radiation. Each different internal configuration corresponds to a different ground state. In principle, if we follow the standard approaches, then the different vacuum states would correspond to different Hilbert spaces at the thermodynamic limit [15]. Yet still, we will see later that the ground states are entangled with each other. The Lagrangian governing the dynamics of the emitted particles is

$$\pounds = \frac{1}{2}\partial^\mu \hat{n}^a_{\mathbf{p}}(\omega)\partial_\mu \hat{n}^a_{\mathbf{p}}(\omega) - V(\hat{n}^a(\omega)). \tag{11}$$

The potential, $V(\hat{n}_{\mathbf{p}})$, is defined as

$$V(\hat{n}_{\mathbf{p}}) = \frac{1}{2}m^2\hat{n}^2_{\mathbf{p}} + \frac{\beta}{3}\hat{n}^3_{\mathbf{p}} + \frac{\lambda}{4}\hat{n}^4_{\mathbf{p}}. \tag{12}$$

We could calculate the ground state by using the condition $\partial V/\partial \hat{n} = 0$. However, due to the spacetime curvature, some kinetic term will still remain, and it cannot be ignored in Equation (11); thus, we have to consider the full version of the Euler–Lagrange equations. The symmetry of the system under exchange of internal configurations, consistent with black hole entropy (exchange of particles), is spontaneously broken when $m^2 < 0$. The signature of the parameter β tells us whether the particles evaporating are bosons or fermions. The spectrum of radiation emerges from applying the Euler–Lagrenge equations over the Lagrangian (11). The information paradox from this perspective suggests that, while the trivial ground state before the formation of the black hole can be represented with a pure quantum state, after the formation of the black hole the vacuum is degenerate, with each vacuum state being represented by each point at the bottom of the potential on Figure 1. Each possible ground state makes a thermal emission of particles if the black hole selects them during the process. The thermal emission then corresponds to a mixed quantum state, while the quantum state corresponding to the internal configuration of the black hole is supposed to be a pure quantum state. This is the source of the information paradox. In the coming section we will look at how this important problem can be solved.

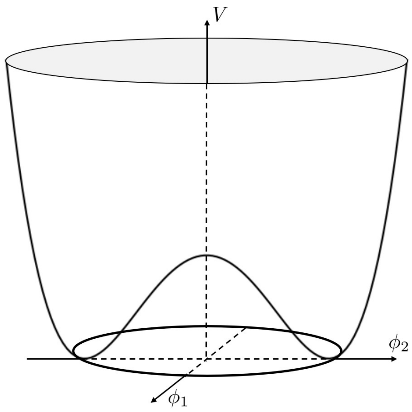

Figure 1. The typical "Mexican hat" potential generated when we have degenerate vacuum states. Each point at the bottom of the potential represents a vacuum state. When the black hole selects one state, it breaks the symmetry spontaneously and then it emits radiation.

5. Mixed Quantum States and the Degenerate Vacuum

The fact that the evaporation process of a black hole corresponds to a spectrum of radiation means that the emitted radiation is independent of the possible internal configurations of the black hole. The degeneracy of the ground state is consistent with this statement. However, more generally, this means that all the possible internal configurations are entangled with each other. Initially, we would be tempted to express the superposition of each possible internal configuration of a black hole with mass M, charge Q, and angular momentum L as

$$|\psi>=\frac{|0>_1+|0>_2+|0>_3+...+|0>_N}{\sqrt{N}}. \tag{13}$$

Here $|0>_i$ corresponds to each possible ground state of the black hole. The quantum state (13) corresponds to a pure state, as can be proved if we construct the density matrix. The only problem with this state is that it suggests that all the possible internal ground states are not entangled. In order to represent the entanglement of the different internal configurations, it is appropriate to express the system as a quantum state of the form

$$|\psi>=\frac{\sum_{i=1}^{N}Tr_{j\neq i}|0>_1\otimes|0>_2\otimes|0>_3\otimes...\otimes|0>_N}{\sqrt{N}}. \tag{14}$$

This quantum state looks more like the state that Hawking imagined, considering that the thermal radiation has to come from tracing out all the possible ground states, except the one which the black hole selects when it breaks the symmetry of the system spontaneously. Expressing the black hole ground state as in Equation (13) or (14) is a matter of convention, and it will not affect the conclusions. We just have to keep in mind that the different ground states are entangled. The pure quantum state represented by state (14) can be expressed through the density matrix

$$\hat{\rho}=\begin{pmatrix}\frac{1}{N} & \frac{1}{N} & ... & \frac{1}{N}\\ \frac{1}{N} & \frac{1}{N} & ... & \frac{1}{N}\\ ... & ... & ... & ...\\ ... & ... & \frac{1}{N} & \frac{1}{N}\end{pmatrix}. \tag{15}$$

We can verify that $Tr(\hat{\rho})=1$ (the trace runs over N entries of the matrix) and that the density matrix is idempotent, which means that the ground state of a black hole, before breaking the symmetry under exchange of particles, is a pure state. Spontaneously breaking the symmetry under exchange of configurations is equivalent to tracing out all the ground states in Equation (15), except the one which the black hole selects. Evidently, any selected ground state, after tracing out all the additional ground states, is a mixed quantum state (non-pure). This is what gives a thermal character to the spectrum of radiation emitted by the black hole. To solve the information paradox, we can argue that each possible ground state corresponds to a mixed quantum state with a probability $p_k = 1/N$, as it is formulated in agreement with Equation (10). Here, N accounts the number of possible ground states, while each ground state has an equal probability of appearing. The states are entangled with each other. The selection of a specific ground state depends on external factors. Among all the possible external perturbations, helping to select some specific direction for the expansion of the universe, we have external gravitational sources, the expansion of the universe itself, and others. Any perturbation generates a preferred direction of selection for some specific ground state. Once this occurs, the emission process starts. In this way, we can describe the evaporation process of a black hole as follows: (1). The black hole is formed and characterized by its mass M, angular momentum L, and charge Q. (2). A degenerate ground state emerges as a consequence of all the internal states consistent with the specific values taken by M, L, and Q. (3). Each ground state has an equal probability $p_k = 1/N$ of emerging as the final ground state. (4). Each ground state is not a pure quantum state, but the total wave function of the system is a superposition of all possible ground states, giving then a final pure (combined) ground state. (5). The selection of a single ground state naturally gives a thermal spectrum corresponding to a mixed quantum state. If we trace out all the ground states, except one, then the resulting density matrix is

$$\hat{\rho}_i = Tr(\hat{\rho})_{j\neq i} = \begin{pmatrix} \frac{1}{N} & 0 & ... & 0 \\ 0 & \frac{1}{N} & ... & 0 \\ ... & ... & ... & ... \\ ... & ... & 0 & \frac{1}{N} \end{pmatrix}. \quad (16)$$

This density matrix naturally represents a mixed quantum state. It is easy to verify that $Tr(\hat{\rho}_i) = 1$, $\hat{\rho}_i^2 \neq \rho_i$, and thus, $Tr(\hat{\rho}_i^2) < 1$. At this point, it is clear that while the black hole quantum state is a pure state as density matrix (15), the Quantum state still represents the thermal spectrum, which is defined in Equation (16); it is a mixed quantum state which does not have a wave function representation.

Recovering Unitarity

It is possible to recover the unitarity of the system if we make a superposition of all the possible spectra of radiation emitted by the black hole. In other words, we can recover the original ground state if we take the particle number operator $\hat{n}_i$ as the order parameter, as was suggested in [12]. Before the formation of the black hole, $<\bar{0}|\hat{n}_{\mathbf{p}}|\bar{0}>= 0$, while after the formation of the black hole we have $<\bar{0}|\hat{n}_{\mathbf{p}}|\bar{0}>\neq 0$. If we consider the black hole evaporation process as a phase transition, then, evidently, the vacuum expectation value of the particle number operator is the order parameter. Considering the vacuum degeneracy, if we sum all the possible values of the order parameter, we should recover the trivial result as explained in [16–20]. The recovery of the original vacuum state from the superposition of all the possible order parameters is a standard result which can be appreciated more from Figure 2. In the same figure, it can be perceived that for every possible vacuum state, such that we can construct the vacuum expectation value of the particle number operator as $<0|\hat{n}_i|0>= n_i$, we have another possible vacuum state with $<0|\hat{n}_j|0>= n_j = -n_i$. Negative particle numbers can be interpreted as the emission of antiparticles. We can recover the vacuum state before the gravitational collapse if we add, over all the possibilities, as follows:

$$<0|\hat{n}_1|0>_1 + <0|\hat{n}_2|0>_2 + ... + <0|\hat{n}_N|0>_N = 0. \quad (17)$$

Another way to interpret this result is by saying that, depending on the external perturbation breaking the symmetry of the black hole, the particle number can take positive and negative values after evaluating the corresponding vacuum expectation values. This is only possible if the black hole emits particles in some situations and antiparticles in others during its particle emission process. We take the emission of antiparticles as a negative number operator after evaluating the corresponding vacuum expectation value. In this way, in general, the black holes can emit particles and antiparticles at any instant. This solves the paradox, because Equation (17) tells us that

$$< \bar{0}|\hat{n}_1|\bar{0} >_1 + ... + < \bar{0}|\hat{n}_N|\bar{0} >_N = < 0|\hat{n}_{bf}|0 >= 0, \quad (18)$$

where the subindex bf stands for "before formation" of the black hole. Then, it is possible to recover all the information for the black hole by summing all the possible outcomes of the order parameter evaluated at their corresponding ground states, taking the order parameter as the particle number operator. When the emitted field represents a particle, then $< \bar{0}|\hat{n}_i|\bar{0} >_i \ > 0$, and when the emitted field represents an antiparticle, $< \bar{0}|\hat{n}_i|\bar{0} >_i \ < 0$. This means that half of the degenerate ground states ($N/2$) correspond to the emission of particles, while the other half ($N/2$) represent antiparticles. Once the black hole selects one among the infinite ground states, then it starts to emit particles. The selection of a single ground state among many requires an external perturbation. The external perturbation could be the presence of another source of gravity, universe expansion itself, or the presence of other virtual pairs in the vicinity. Normally, the origin or source of the perturbation is not so relevant, because its influence has to be considerably small once the conditions for symmetry breaking are reached. Figure 3 illustrates this issue, where virtual pairs are generated in the neighborhood of the black hole. For the emission process to occur, there must be a special orientation for the pair with respect to the event horizon; in other words, one particle from the pair can enter the event horizon when the right orientation, with respect to the event horizon, is established. Particular orientations of the virtual pairs at certain instants are defined by external perturbations.

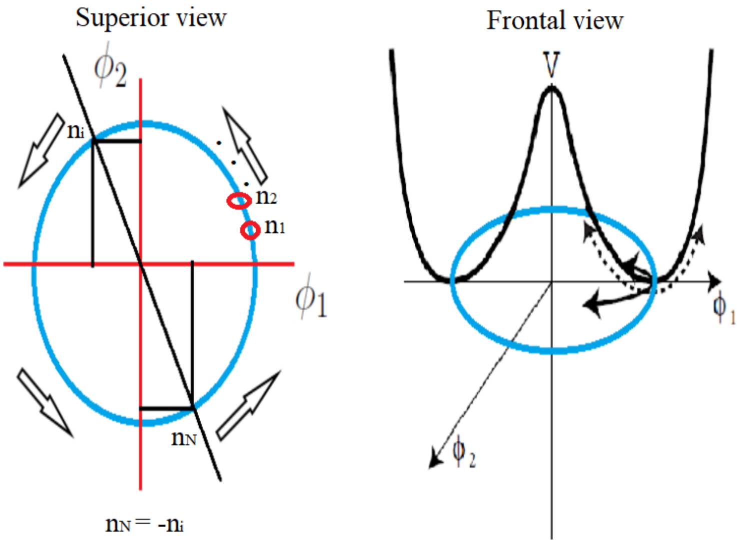

Figure 2. Typical potential when there is a vacuum degeneracy. In such a case, the system selects one among the N possible ground states. Before the symmetry is broken, for each possible vacuum $< 0|\hat{n}_i|0 >$, we have a diametrically opposite vacuum obeying $< 0|n_N|0 >= - < 0|\hat{n}_i|0 >$. This can be interpreted as the emission of antiparticles when the appropriate vacuum state is selected during the breaking process.

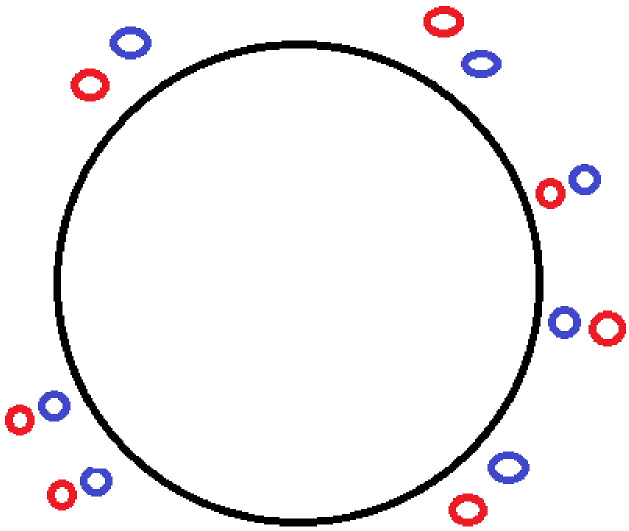

Figure 3. Black Hole evaporation process. Virtual pairs with particles (blue circles) and antiparticles (red circles) are generated in the neighborhood of the black hole. Their orientation with respect to the black hole is random. From the perspective of the mechanism of spontaneous symmetry breaking, whether or not one of the particles in the pair enters the event horizon depends on external perturbations.

6. Conclusions

In this paper, we have demonstrated that it is possible to solve the black hole information paradox if we consider the process as a consequence of the breaking symmetry under internal exchange of configurations spontaneously. From this perspective, the particle operator number $\hat{n}_{\mathbf{p}}$ is the order parameter of the system, which is trivial before the formation of the black hole (zero vacuum expectation value) and non-trivial after the formation of the black hole (non-zero vacuum expectation value) and once one of the degenerate vacuum states is selected during the process. Even still, the whole quantum state representing the black hole corresponds to a pure quantum state which considers all possible (degenerate) vacuum states. Selecting one of the ground states is equivalent to tracing out all the other vacuum states, and this is why Hawking radiation has a thermal nature, generating a mixed quantum state (non-pure). The information for the black hole is recovered when we sum all the possible order parameters emerging from the breaking of the symmetry, under internal configurations, spontaneously. This brings another physical consequence, suggesting that black holes not only emit particles, but also antiparticles. The present formulation solves the information paradox of black holes. The emission of antiparticles could be noticed through the electromagnetic spectrum that they would generate after suffering annihilation when they meet ordinary particles. Analyzing how strong this signature can be is be a matter of evaluation, and discussion will follow in a forthcoming paper.

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